Conformal Field Theory and Two Dimensional Critical Phenomena

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Abstract

We explore the representations of the underlying infinite dimensional symmetry algebra in two dimensional quantum field theory. This provides a window to study the algebra of local fields which is central to the description of two dimensional critical phenomena that emerges in the scaling limit of certain statistical systems on a two-dimensional lattice. Unitary minimal models are then constructed from such representations and the first few series of the set of minimal theories are shown to correspond to some well-known statistical systems. The systems studied in particular are the Ising and the tricritical Ising models. Potential interactions between Ising fermions and bosons in two dimensions occurring within the framework of specific condensed matter models at criticality are reviewed using arguments from conformal invariance.

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Contents

1 Introduction

Two dimensional conformal field theory (CFT) is a quantum field theory endowed with conformal symmetry. Conformal invariance arising at critical fixed points was first put forward by Polyakov in 1970 [21]. The understanding of conformal field theories has proven to be a relatively arduous exercise in many areas of theoretical physics. It was not until more than a decade later that the field would received serious consideration through the seminal work of Belavin, Polyakov and Zamolodchikov on conformal field theory in two dimensions and its applications in statistical physics [2]. Since then conformal field theory unfolded very rapidly in many new direction and was found to provide an accurate description of critical phenomena. Aside to their vital role in understanding the classification of fixed points of the renormalization group (RG) and providing explanations for the occurrence of scale-invariant theories [2, 6, 7], CFTs proved prominently crucial in the development of string theory, for example, conformal invariance of the world sheet is necessary for hindering the appearance of ghosts degrees of freedom which lead to non-positive definition of probabilities in the quantum theory. Furthermore CFTs are central for studying quantum gravity via the AdS/CFT correspondence [1, 38]. Since its establishment conformal field theory has been receiving extensive inputs from pure mathematics. An important one being the Kac formula [1, 2, 3, 4, 6, 8] which was explicitly proven in Ref. [30] and is extensively employed in the study of minimal models which we shall encounter later on.

When a system tends towards criticality the physical description is outlined by its correlation length which is much greater than all microscopic scales. Being independent of microscopic scales allows many systems to look similar at criticality. This is known as the principle of universality [1]. The critical exponents associated to all universal quantities are related to the correlation length. The latter diverges at criticality indicating that the system is loosing its only scale and acquires scale symmetry defined globally. If interactions are local then the system bears local scale symmetry and angle preserving - length altering transformations emerge which are the conformal transformations. In this thesis, we shall essentially work with systems which act primarily via local interactions. In such cases, as described by the authors of Ref. [1, 6] for homogeneous and isotropic systems, conformal symmetry is bound to follow from scale invariance such that the classification of the renormalization group fixed points[1] is similar to the construction of CFTs. A concrete demonstration of this can be found in Zamolodchikov' s paper [7]. Generally speaking, conformal transformations are simply dilations by a scaling factor which is dependent on positions (non-rigid or local dilations).

Even after enhancement by conformal invariance, we still have a finite number of parameters required to specify a conformal transformation in an arbitrary number of dimensions d which is $\frac{1}{2}(d+1)(d+2)$. The result of this limited number of parameters restrict the amount of information that can be obtained from correlation functions. This restriction is however lifted when the number of dimensions is two, due to the infinite collection of well-defined non-rigid (local) transformations which are essentially local dilations. The number of parameters specifying local conformal transformations is therefore infinite which explains why conformal invariance in two dimensions is so successful. The details of this argument are presented in subsection (2.2). This allows us to construct correlations functions and derive valuable information (such as the critical exponents) as

[1] A fixed point in a field theory is one which has the symmetry $x^\mu \to \alpha x^\mu$ in all scales, where x^μ are coordinates of space and α is just a scaling parameter [6].

we shall see.

Outline of thesis

In Chapter 2, the general features of conformal field theory in an arbitrary number of dimensions are introduced before considering cases where the number of dimensions is reduced to two. From the operator product expansion formalism we introduce the central charge and show how the c-value characterizes models at criticality. We focus specifically the massless free fermion case which has the c-value of the Ising model [8]. The explicit derivation of the c-theorem is provided and shown to be accurate only up to a perturbation using RG flow methods [7] in the final section of this chapter. In chapter 3, we study the structure of conformal field theory from the point of view of radial quantization which provides an efficient way of ordering operators later in the chapter. The central extension to the Witt algebra $\mathfrak{W}$, the Virasoro algebra $\mathfrak{V}$, is explicitly derived. We discuss the further constraints (aside to those imposed by conformal symmetry) imposed by the operator algebra on the three- and four-point correlation functions. The representations of the Virasoro algebra are introduced and discussed in chapter 4 as we want to provide a physical meaning to the underlying symmetries present in our theories. This allows to then construct the minimal models. In chapter 5, the Landau-Ginzburg theory is introduced which provides a different framework based on an effective Lagrangian approach [1] for studying minimal models. In the final chapter, we discuss an application of the minimal theory $\mathcal{M}(5,4)$ which lies in the universality class of the tricritical Ising model to the phenomenon of emerging spacetime supersymmetry motivated by the recent numerical analysis [28] in $d = 1+1$. A plan for future work is then later suggested as part of the final remarks.

2 Aspects of conformal field theory

Conformal transformations, also known as conformal mappings, represent the set of general coordinate transformations $x^\mu \to x'^\mu$ which are invertible mappings and orientation preserving[2] [1]. From a very general point of view, one can ask what does a conformal mapping actually means? Considering for example an analytic function; the latter is conformal at any point where it has a non-vanishing derivative. On the contrary, any conformal mapping of a complex variable which has a continuous partial derivative is analytic. These conformal transformations in d dimensions can be described by the noncompact group[3] $SO(d+1,1)$. The symmetries associated to this group and further techniques necessary for understanding two dimensional conformal field theories are introduced in this chapter.

2.1 Algebra of conformal symmetry

From the definition of conformal transformation above, the line element is written down up to scale:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \tag{1}$$

where $\Lambda(x)$ is the conformal factor and represents some function of x. It can be noted how locally a conformal transformation accounts for rotation and dilation such that it

[2] On a more formal note, one can also say that a conformal mapping represents a bijective homomorphism [18] (Def. 1.2).

[3] Let' s consider an element A where $A \in \mathbb{R}^N$ is compact $\iff$ A is closed and bounded (which means it can be fitted inside a sphere surrounding the origin). Noncompactness can be therefore (informally) referred to as not fitting into the aforementioned sphere [18].

evidently forms a group. The special case when $\Lambda(x) = 1$ corresponds to the subset which is the Poincaré group [1]. We now consider an infinitesimal transformation of the form $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$, where $\epsilon(x)$ is a small parameter (i.e: terms proportional to $\mathcal{O}(\epsilon^2)$ or higher are disregarded). The metric $g_{\mu\nu}$ is found to take the following form

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu. \tag{2}$$

The derivation of the above equation is not as trivial as one might think, hence we explicitly work out the details by first considering a Taylor expansion of $g'_{\mu\nu}(x')$ which is

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x) + \partial_\lambda (g'_{\mu\nu})\epsilon^\lambda + \mathcal{O}(\epsilon^\lambda) + \dots \ . \tag{3}$$

A covariant transformation of $g'_{\mu\nu}(x')$ is

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(x) = (\delta^\alpha_\mu - \partial_\mu \epsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \epsilon^\beta)\, g_{\alpha\beta} \\ &= g_{\mu\nu} - \delta^\alpha_\mu(\partial_\nu \epsilon^\beta) g_{\alpha\beta} - \delta^\beta_\nu(\partial_\mu \epsilon^\alpha) g_{\alpha\beta} + \mathcal{O}(\epsilon). \end{aligned}$$

The dummy indices on $\epsilon(x)$ can be replaced by λ and the $g_{\alpha\beta}$ can be contracted by the δ^α_μ and δ^β_ν such that

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - (\partial_\nu \epsilon^\lambda) g_{\mu\lambda} - (\partial_\mu \epsilon^\lambda) g_{\lambda\nu} + \dots \ . \tag{4}$$

We can now equate (1) and (2), using the product rule $\partial_\mu(\epsilon^\lambda g_{\mu\lambda}) = (\partial_\mu \epsilon^\lambda) g_{\mu\lambda} + (\partial_\mu g_{\mu\lambda})\epsilon^\lambda$ in eq. (2) and symmetrizing over the indices μ and ν, one obtains the following

$$g'_{\mu\nu}(x') = g_{\mu\nu} - \partial_\mu \epsilon_\nu + \frac{1}{2}[\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}]\epsilon^\lambda - \partial_\nu \epsilon_\mu + \frac{1}{2}[\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\nu\mu}]\epsilon^\lambda, \tag{5}$$

where the quantities in the square brackets can be recognized as being the Christofel symbols such that one can rewrite[4] eq. (5) as

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu, \tag{6}$$

where $\nabla_\mu \epsilon_\nu = \partial_\mu \epsilon_\nu - \Gamma_{\mu\nu\lambda}\epsilon^\lambda$ and similarly when $\mu \leftrightarrow \nu$. However since we are working on flat space, the Christofel symbols vanish and we are left with eq. (2) which completes the proof.

Under the infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu(x)$ we can express the conformal factor as $\Lambda(x) = 1 + f(x)$, such that for a conformal mapping we demand that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}, \tag{7}$$

where $f(x)$ is determined by contracting both sides eq. (7) with $g^{\mu\nu}$ such that

$$f(x) = \frac{2}{d}\partial_\rho \epsilon^\rho, \tag{8}$$

where $d = g^{\mu\nu} g_{\mu\nu}$ corresponds to the dimensions of the manifold. Substituting eq. (8) in (7), one obtains the conformal Killing equations [18]. In order to understand the conformal transformation in d dimensions, we need to solve eq. (7), the details can be found in

[4] The primes on $g_{\mu\nu}$ in eq. (5) does not appear when equating (1) and (2), this is because the difference between the derivatives of the primed and unprimed $g_{\mu\nu}$ is only of order $\mathcal{O}(\epsilon)$.

Ref. [1]. This leads to the following explicit relation for conformal transformations in an arbitrary number of dimensions

$$(d-1)\partial^2 f = 0. \tag{9}$$

From the above equation we can look at different cases corresponding to different values of d. For example $d = 1$ is trivial since there are no restrictions on f such that any smooth transformation in one dimension is conformal. In two dimensions, eq. (7) represents the Cauchy-Riemann equations for different values of μ and ν with solutions given as holomorphic (or antiholomorphic) functions generating conformal transformations. This will be further discussed later on as it is the most interesting for studying systems in statistical physics. Finally when $d \geq 3$, the general solution to (7) can be expressed as

$$\epsilon_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + \lambda x^\mu - b^\mu x^2 + 2(b\cdot x)x^\mu, \tag{10}$$

where $b_{\mu\nu} = -b_{\nu\mu}$ is antisymmetric and where $a_\mu, b_{\mu\nu}, \lambda$ and b^μ which corresponds to translations, rotations, scale transformations and special conformal transformations respectively. λ is an arbitrary scalar. When $d = 4$, one can note that the total number of parameters for the above set of transformations is 15. This is easily obtain by counting the number of degrees of freedom[5] associated to $a_\mu, b_{\mu\nu}, \lambda$ and b^μ which are **4, 6, 1, 4** respectively. A complete derivation of eq. (10) can be found in Ref. [1] (Pg. 96-97).

So far we have been looking infinitesimal conformal transformation generated by eq. (7). Those transformations can also be defined for every points on the manifold $\mathbb{R}^n$ leading to global conformal transformations [1]. The expressions for the finite transformations generating translations, rotations and scale transformations are written in a form which is familiar to the first three terms in eq. (10). The last transformations, the special conformal transformations, however are expressed differently. This can shown by the following construction which is the composition of the global conformal mapping of inversion $x^\mu \to x^\mu/x^2$, followed by a translation (infinitesimal) b^μ and followed by a further inversion [17]

$$x^\mu \to \frac{x^\mu}{x^2} - b^\mu \to \frac{\frac{x^\mu}{x^2} - b^\mu}{(\frac{x^\mu}{x^2} - b^\mu)^2} = \left.\frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}\right|_{b^\mu\to 0} = x^\mu - b^\mu x^2 + 2(b\cdot x)x^\mu, \tag{11}$$

where the far right hand side corresponds to the infinitesimal special conformal transformations (which matches with the last two terms of eq. (10) when we take $b^\mu \to 0$ and Taylor expand). The above global transformations form the conformal group where the generators for translations, Lorentz rotations, scale transformations and special conformal transformations are $P_\mu, L_{\mu\nu}, D$ and K_μ respectively. This forms the 15-parameter group which is an extension of the 10-parameter Poincaré group. The conformal algebra for the 15-parameter group therefore closes as shown by the following commutation relations [1]

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\rho} + \eta_{\nu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})$$

$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)$$

$$[P_\mu, P_\nu] = 0$$

$$[K^\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu})$$

[5] The number of degrees of freedom is obtain by counting the number of components which are: 4 for each of the vectors a_μ and b_μ, 1 for the scalar λ and $(16-10) = 6$ for the antisymmetric tensor $b_{\mu\nu}$, where $10 =$ number of symmetric components. Or explicitly, one can write down $\frac{(d+1)(d+2)}{2}$ for the number of parameters.

$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \tag{12}$$
$$[D, K_\mu] = -iK\mu$$
$$[D, P_\mu] = iP_\mu$$
$$[D, D] = [L_{\mu\nu}, D] = 0$$
$$[K_\mu, K_\nu] = 0,$$

where the first three commutations relations corresponds to the algebra of the Poincaré group while the subsequent algebras correspond to the extension of this bosonic spacetime symmetry. The whole set of commutation relation corresponds to algebra of the conformal group in $d \geq 3$.

2.2 Conformal invariance in two dimensions

The group of conformal symmetry, as discussed in the previous section is finite in $d \geq 3$ but this is not the case in two dimensions as we shall see. A much larger class of solutions is obtained from only two differential conditions on two functions when $d = 2$ provided $\epsilon^\mu(x)$ is conformal. If $(\mu, \nu) = (1, 2)$ eq. (7) becomes the Cauchy-Riemann equation for the holomorphic functions [3]

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \qquad \partial_1\epsilon_2 = -\partial_2\epsilon_1. \tag{13}$$

The antiholomorphic counterparts are defined by simply alternating the signs in eq. (13). It can therefore be more convenient to have the domain to be the entire complex plane $\mathbb{C}$. The solutions for eq. (13) are thus naturally expressed as

$$z = x_1 + ix_2, \qquad \bar{z} = x_1 - ix_2, \tag{14}$$

where $z, \bar{z} \in \mathbb{C}$ such that we can define $w(z, \bar{z}) = \epsilon_1(x_1, x_2) + i\epsilon_2(x_1, x_2)$ and $\bar{w}(z, \bar{z}) = \epsilon_1(x_1, x_2) - i\epsilon_2(x_1, x_2)$ which corresponds to the holomorphic and antiholomorphic Cauchy-Riemann equations respectively expressed as

$$\partial_{\bar{z}}w(z, \bar{z}) = 0, \qquad \partial_z\bar{w}(z, \bar{z}) = 0. \tag{15}$$

Hence the coordinates in eq. (14) shall be treated as two independent complex variables instead of complex conjugates [2].

The solutions of eq. (15) generally represent a collection of holomorphic and anti-holomorphic mappings as discussed above and we know that any analytic mapping of the complex plane onto itself is orientation preserving, that is conformal. The conformal group in two dimensions, $\mathcal{G}$ is the group of all analytic maps endowed with group multiplication representing composition of maps such that $\mathcal{G} = \Gamma \otimes \bar{\Gamma}$ defined on $\mathbb{C}^2$, where Γ represents the set of all the holomorphic coordinates z while $\bar{\Gamma}$ accounts for all the antiholomorphic counterparts. Infinitesimal conformal transformations of the group Γ are given by the Laurent series which consist of an infinite number of coefficients required for specification of all functions analytic in some neighborhood. It is exactly this infinite number[6] of degrees of freedom that is responsible for the huge amount of information prevailing within the framework of conformally invariant quantum field theories in two

[6] In this language, we are saying that the set of conformal (anti)holomorphic maps of the complex plane is infinite dimensional however one can reiterate the latter in a more precise statement which is that we have an infinite dimensional Lie algebra which is closely related to the algebra of conformal symmetry in two dimensions; the Witt algebra, or its central extension of the latter, the Virasoro algebra [18].

dimensions.

So far we have simply been considering local conformal transformations, which means that no constraints were imposed demanding that the transformations are well-defined at every points and be invertible on the Riemann sphere defined as $\mathbb{S}^2 := \mathbb{C} \cup \infty$ [3]. The only set of transformations satisfying these constraints are the global conformal transformations represented by the special conformal group identified by the following mapping

$$f(z) = \frac{az+b}{cz+d}, \qquad \text{with} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}), \tag{16}$$

where $\mathrm{SL}(2, \mathbb{C})$ refers to the special linear group satisfying the constraint $ad - bc = 1$ which actually identifies the $f(z)$ as the Mobius group[7] after modding out the discrete group $\mathbb{Z}_2$. The map $f(z)$ should be free of any essential singularity such that the only singularities that are assumed are poles, which means f can be written as a function of polynomials where the denominator and numerator have different zeros (further details can be found in [1] eq. (5.14)). The only possibilities for f to be invertible is for both the denominator and the numerator to be linear functions. We consider some transformations of f to try and justify eq. (16) such as translations and rotations where $f(z) = z + a$ and $f(z) = bz, |b| = 1$ respectively with a, b being constants. If we now lift the restriction that $|b| = 1$ and have instead $b \in \mathbb{R}$, then $f(z) = bz$ shall correspond to a dilation [22]. Formally translations, rotations and dilations are all conformal transformations (i.e; angle preserving transformations). Another conformal transformation is the inversion, $f(z) = \frac{1}{z}$ that we briefly talked about in the previous subsection. It does not seem very convincing that an inversion is a symmetry due to the singularity at $z = 0$. This has actually been taken care of when we defined our space to be the complex plane $\mathbb{C}$ and a point at $z = \infty$. Hence upon combining all these transformations, one can see how they form the group of global conformal transformations given in eq. (16).

2.2.1 Generators of the classical conformal algebra

In many instances when describing physical systems it is useful to invoke local properties rather than global properties. This allows one to find the algebra of the generators by considering an infinitesimal Laurent expansion (of the group Γ we introduced above) in the neighborhood of $z = 0$. We note that the properties for holomorphic sector are similar to those of the antiholomorphic sector, hence we shall deal solely with the former. The infinitesimal transformations are

$$z = z + \epsilon(z) \quad \text{where} \quad \epsilon(z) = \sum_{n \in \mathbb{Z}} b_n z^{n+1}. \tag{17}$$

We consider this infinitesimal mapping on a spinless and dimensionless field on $\mathbb{C}$ hence the latter transform as [1]

$$\phi'(z', \bar{z}') = \phi(z', \bar{z}') - \epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z', \bar{z}'), \tag{18}$$

where we Taylor expanded the field in the vicinity of $z = 0$. It is convenient to express eq. (18) as

$$\delta\phi = \phi'(z', \bar{z}') - \phi(z', \bar{z}') = -\epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z', \bar{z}'), \tag{19}$$

[7] The Mobius group is isomorphic to the group PSL $(2, \mathbb{C}) :=$ SL $(2, \mathbb{C})/\mathbb{Z}_2$. The mappings f are thus known as projective mappings [1].

where the left hand side of eq. (19) can be expressed as

$$\sum_{n\in\mathbb{Z}}[b_n l_n \phi(z,\bar{z}) + \bar{b}_n \bar{l}_n \phi(z,\bar{z})], \tag{20}$$

where b_n and $\bar{b}_n$ are just the coefficients of in the Laurent series expansion and where we define the generators[8] as

$$l_n := -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n := -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{21}$$

The above relations can be used to construct the conformal algebra which closes according to the following commutation relations as shown

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{m+n} \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{m+n} \\ [l_n, \bar{l}_m] &= 0. \end{aligned} \tag{22}$$

The algebra in eq. (22) is commonly known in CFT literatures as the Witt algebra which we shall denote as $\mathfrak{W}$ (and $\bar{\mathfrak{W}}$ for the antiholomorphic sector, i.e: the second line of (22)). The conformal algebra is therefore represented as the direct sum of holomorphic and antiholomorphic component $\mathfrak{W} \oplus \bar{\mathfrak{W}}$. The latter is infinite however for distinct values of $n = \{-1, 0, 1\}$, we can form a subalgebra of $\mathrm{SL}(2,\mathbb{C})$ which is finite such that it consists of the projective conformal transformations (16). Such algebra can be written as $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. From the definition (21) and for the designated values of n we have the following generators $\{l_{-1}, l_0, l_1\} = \{-\partial_z, -z\partial_z, -z^2\partial_z\}$ respectively corresponding to translations, dilations and special conformal transformations on the complex plane. The linear combinations $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ respectively generates dilations and rotations on the real surface. If we denote the eigenvalues of the two operators l_0 and $\bar{l}_0$ as h and $\bar{h}$ respectively then we can define the scaling dimension Δ and spin s of the state as $\Delta = h + \bar{h}$ and $s = h - \bar{h}$ respectively [3], where both h and $\bar{h}$ are real.

2.2.2 Correlation functions

The mathematical details presented in the previous subsection will be used to discuss correlation functions which represent the physically measurable quantities in field theories. It will be demonstrated in the next section how the correlators constructed exclusively from primary fields satisfy the conformal Ward identities. We now proceed to derive the correlators from the constraints imposed by conformal invariance [1, 3] such that under the transformation $z \to g(z)$ and $\bar{z} \to \bar{g}(\bar{z})$, a quasi-primary field transforms as

$$\phi'(g,\bar{g}) = \left(\frac{dg}{dz}\right)^{-h}\left(\frac{d\bar{g}}{d\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}), \tag{23}$$

which essentially represents a generalization of field transformations under a change of coordinate; $\phi(x) \to \phi'(x') = |\partial x'/\partial x|^{-\frac{\Delta}{d}}\phi(x)$, where Δ is the scaling dimension of $\phi(x)$ as we saw earlier and d represents the number of dimensions [3]. The conformal weight of ϕ expressed as $(h,\bar{h})$ are real-valued quantities. Under a small perturbation close to the identity, that is $g = z + \epsilon(z)$ and $\bar{g} = \bar{z} + \bar{\epsilon}(\bar{z})$, an infinitesimal change in the quasi-primary field reads

$$\delta_{\epsilon,\bar{\epsilon}} = -(h\phi\partial_z\epsilon + \epsilon\partial_z\phi) - (\bar{h}\phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\phi). \tag{24}$$

[8]We can say that the Lie algebra of the group Γ we introduced earlier coincides with the algebra of differential operators.

This is easily derived by substituting the perturbed fields $g(z)$ and $\bar{g}(\bar{z})$ in (23) and Taylor expanding leads to (24). Any field transforming under (24) is known as a primary field [1]. The difference between a quasi-primary and a primary is that the former transforms under an element of $\mathrm{SL}(2,\mathbb{C})$ which is strictly part of conformal transformations. This restriction is lifted upon dealing solely with primary fields. A field which does not fall in the category of primary fields is known as a secondary field and generally applies to derivatives of a primary fields.

The correlation function for n primary fields ϕ_i in two dimensions then follows from (23), where the conformal transformations are expressed as

$$\langle \phi_1(g_1,\bar{g}_1)...\phi_n(g_n,\bar{g}_n)\rangle = \prod_{i=1}^{n} \left(\frac{dg}{dz}\right)^{-h_i}_{w=w_1} \left(\frac{d\bar{g}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}_i=\bar{w}_i} \langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle \,. \tag{25}$$

Under the constraints imposed by global conformal invariance, this relation above allows us to fix the two- and three-point correlation functions. This procedure is demonstrated in Ref. [3] (Ch. 2). We can write the two and three point functions respectively as

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}}, \tag{26}$$

where $z_{12} = (z_1 - z_2)$ and $\bar{z}_{12} = (\bar{z}_1 - \bar{z}_2)$. C_{12} is just a constant to be determined by normalizing the fields. The above is therefore valid for the holomorphic conformal dimension $h_1 = h_2 = h$ and similar for the antiholomorphic dimension. The two-point function vanishes if the conformal dimensions are different. Similarly the three-point function takes the following form

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \times \begin{pmatrix}\text{antiholomorphic}\\ \text{sector}\end{pmatrix}. \tag{27}$$

Eq. (26) and (27) will be central in understanding the operator product expansions in section (2.2.4) as the latter will bring us to consider the operator algebra formalism which imposes extra constraints beyond that of global conformal invariance on the two- and three-point functions. These ingredients are fundamental for understanding the conformal bootstrap [1, 2, 6] introduced by Polyakov [21] and independently by Kadanoff [13, 14] in order to appreciate the implications of CFTs in statistical physics systems. Extending the above approach to the four-point function turns out to be a non-trivial task and requires some additional information.

In deriving eq. (26) and (27) in an arbitrary number of dimensions d as demonstrated in Ref. [1] (Ch. 4), functions which are left unchanged under all known conformal transformations are introduced. These are referred to as conformal invariants. The latter are constructed by invoking the simplest form of conformally invariant cross-ratios (or anharmonic ratios) [1, 3, 23] which read

$$\frac{r_{12}r_{34}}{r_{13}r_{24}} \quad \text{and} \quad \frac{r_{12}r_{34}}{r_{23}r_{34}}, \tag{28}$$

where $r_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$. Hence this implies that we can explicitly calculate correlation functions, however complications rapidly creep up when we extend the above procedure to the four-point function. In d dimensions, global conformal invariance is not enough and the four-point (and n-point) functions show dependence on the cross-ratios (28) [1, 23].

This makes it difficult to explicitly write something that is in a similar form to eq. (26) or (27) for a four-point correlation function. We can argue that the four-point function can be expressed as a product of some simple linear functions and the cross-ratios invariant under conformal symmetry when the number of dimension is two. This is because of the large reduction of the number of independent cross-ratios [1]. This reduction is however not as large as for the three-point function which is completely fixed as we' ve seen but it is nonetheless sufficient for the moment. As the four points are constrained on the same plane, we can therefore write down explicitly the additional linear relations which represent the six possible cross-ratios. we shall quote the latter from Ref. [1] to be

$$x = \frac{z_{12}z_{13}}{z_{13}z_{24}}, \qquad 1-x = \frac{z_{14}z_{23}}{z_{13}z_{24}}, \qquad \frac{x}{1-x} = \frac{z_{12}z_{34}}{z_{14}z_{23}}, \tag{29}$$

plus the inverses. Generally the four-point correlation function can be expressed as

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle = f(x,\bar{x}) \prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}, \tag{30}$$

where $f(x,\bar{x})$ represents some form of the linear functions as we discussed previously and $h = \sum_{i=1}^{4} h_i$ and $\bar{h} = \sum_{i=1}^{4} \bar{h}_i$. Out of the four points one can fix three of these points, say z_1, z_2, z_3 for example, via global conformal invariants such that $z_1 = 0, z_2 = 1, z_3 \mapsto \infty$. The last point x_4 will depend on the anharmonic ratios such that we can send $x_4 = x$ which fixes the general expression[9] for the four-point correlation function.

2.2.3 The conformal Ward Identity

In this subsection the conformal Ward identities are introduced. Ward identities are generally objects satisfied by correlation functions as a consequences of an underlying symmetry of a theory and are obtained by understanding the behavior of n-point correlation functions under a conformal transformation [3]. However before examining the consequences of conformal symmetry for the n-point functions on the complex plane $\mathbb{C}$, it is important to consider transformations which are locally conformal. This forces us to consider briefly the following action defined in terms stress-energy tensor $T_{\mu\nu}$ subjected to an infinitesimal transformation which we write as

$$\delta S = \int d^d x T_{\mu\nu} \beta^{\mu,\nu}, \tag{31}$$

where $\beta^{\mu,\nu}$ represents a general infinitesimal transformation and $T_{\mu\nu}$ is conserved and symmetric as implied by invariance of S under translations and rotations (Lorentz) [19]. Furthermore if S is invariant under scale symmetry then $T_{\mu\nu}$ can be made traceless. From eq. (7), we understand that under infinitesimal conformal symmetry $T_{\mu\nu}$ is traceless which is indicative of invariance under conformal symmetry. The reverse is untrue due to the fact that $\beta^{\mu,\nu}$ is not arbitrary (and it has to be as it is precisely the different forms the latter can have that will allow us to derive the conformal Ward identity). Hence we see that the full conformal invariance is a consequence of scale and rotational invariance [1, 3, 19].

[9] The full four-point correlation function and a more general form of $f(z,\bar{z})$ for n-point functions can be obtained via differential equations where the monodromy conditions are imposed. Monodromy invariance allows constraints to be imposed such that the physical correlators are single-valued [1] (Ch. 8, 9).

We now proceed to the conformal Ward identity by first considering the correlation function $\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle$ defined on $\mathbb{C}$. An infinitesimal transformation $z \to z' = z + \beta(z)$ on the points $\{z_i\}$ is made inside a region $\mathcal{M}$ where $\mathcal{M} \subset \mathbb{C}$ (i.e: a region lying inside the complex plane [19]) by defining a contour C which contains all the points $\{z_i\}$ and C is found inside region $\mathcal{M}$. This allows the transformation to be conformal within C leading to infinitesimal discontinuities within C. Hence eq. (31) can be rewritten[10] in terms of complex coordinates as

$$\delta S = \frac{1}{2\pi i} \int_C \{dz \beta(z) T(z) - d\bar{z} \beta(\bar{z}) T(\bar{z})\} \tag{32}$$

. Taking in order to account for extra terms coming from path integral definition of S [1] (Pg. 104, Sec. 4.3), we can express the explicit change in the correlation function under local conformal transformation as

$$\delta \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle = \frac{1}{2\pi i} \int_C dz \beta(z) \langle T(z)\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle + \text{c.c}, \tag{33}$$

where c.c stands for complex conjugate. When $\beta(z) = \lambda(z - z_1)$, corresponding to a combined rotation and scale dilation, the variation in ϕ_1 is given as $\delta\phi_1 = (h_1\lambda + \bar{h}_1\bar{\lambda})\phi_1$. Substituting all this information in (33) and comparing coefficients of λ and $\bar{\lambda}$ lead to

$$\frac{1}{2\pi i} \int_C dz(z - z_1) \langle T(z)\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle = h_1 \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle + ... \ . \tag{34}$$

If $\beta(z)$ is now a constant then $\delta\phi_i \propto \partial_{z_i}\phi_i$ up to first order in β such that

$$\frac{1}{2\pi i} \int_C dz \langle T(z)\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle = \sum_i h_i \partial_{z_i}\phi_i \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle . \tag{35}$$

From Cauchy' s theorem, we can then combine (34) and (35) in order to determine the correlation function of the fields ϕ_i with $T(z)$ over discrete regions (the contours we established earlier [3]) specified by the singularities (i.e: we ignore the part of the series where the function is regular at $z = z_i$). The conformal Ward identity is therefore expressed as

$$\langle T(z)\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle = \sum_i \left[\frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i}\partial_{z_i} \right] \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)...\rangle . \tag{36}$$

where h_i represent the dimensions of the primary fields ϕ_i. The primary fields together with secondary fields associated to them (which are the descendants as we shall see later on) form a closed operator algebra. The whole set of fields $\{\Phi_i\}$ constitute a conformal family [1, 2, 3] which can be represented as $\{\Phi_i\} = \bigoplus_n [\phi_n]$. This will be crucial when deriving the minimal models later. An equivalent definition of (32) prevails for the antiholomorphic sector as well.

[10] Obtaining eq. (32) from eq. (31) require a few steps which we are going to briefly discuss. We shall consider a vector F^μ where the divergence of F^μ is integrated within $\mathcal{M}$ which we defined to be within the complex plane $\mathbb{C}$ bounded by the contour $C = \partial\mathcal{M}$. From Gauss theorem, one has $\int_\mathcal{M} d^2x\, \partial_\mu F^\mu = \int_{\partial\mathcal{M}} dn_\mu F^\mu$, where dn_μ represents the small line element of C with the normal of dn_μ pointing outwards orthogonal to C. Setting $dn_\mu = \varepsilon_{\mu\rho} ds^\rho$ such that ds^ρ is parallel to C. It is easier to work with an integration running anti-parallel to C rather than one which perpendicular to it. Hence we now have a usual contour integral such that $\int_\mathcal{M} d^2x\, \partial_\mu F^\mu = \int_{\partial\mathcal{M}} \{dz\, \varepsilon_{\bar{z}z} F^{\bar{z}} + d\bar{z}\, \varepsilon_{z\bar{z}} F^z\} = (1/2i) \int_{\partial\mathcal{M}} \{dz\, F^{\bar{z}} - d\bar{z}\, F^z\}$ which is the form in which eq. (32) is presented. $\varepsilon_{\mu\nu}$ is the 2 by 2 matrix in Eq. (5.8) Pg. 113 [1] which we used above.

2.2.4 The operator product expansion and the central charge

The operator product expansion (OPE) states that the product of the two fields defined locally within close proximity of each other can be represented by a sum of local operators up to arbitrary accuracy. The terms present in the OPE are essentially singular as reflected in our derivation of the conformal Ward identity in (36). Hence the OPE of the stress-energy tensor with a single primary field $\phi(z, \bar{z})$

$$T(z)\phi(z_1, \bar{z}_1) \sim \left[\frac{h_1}{(z - z_1)^2} + \frac{1}{z - z_1}\partial_{z_1}\right]\phi_1(z_1, \bar{z}_1), \tag{37}$$

where the symbol $\sim$ indicates equal up to non-singular terms [1]. In a classical theory invariant under conformal symmetry the transformation of the stress-energy tensor under an infinitesimal change, $z = z + \beta$ reads

$$\delta_\beta T = 2T(\partial\beta) + \beta(\partial T), \tag{38}$$

where the holomorphic conformal dimension, $h = 2$ for the stress-energy tensor. A similar expression exists for the antiholomorphic sector. The above equation relation is easily derived by substituting eq. (36) in eq. (33) and computing the variation of a single primary field ϕ_1 for example, one therefore has $\delta_\beta\phi_1 = \phi_1(\partial\beta) + \beta(\partial\phi_1)$ which is consistent with eq, (19) where $h_1 = 1$ for a primary field[11].

Under scale transformations and translations, we have for the holomorphic primary field $T(z)$ that $\langle T(z)\rangle = 0$, similarly for antiholomorphic sector (i.e: $\langle T(\bar{z})\rangle = 0$) in the quantum theory. The stress-energy tensor $T(z)$ is a symmetric and traceless representation of an energy density with scaling dimensions $\Delta = 2$ and spin $s = 2$ such that $h = \frac{1}{2}(\Delta + s) = 2$. Hence the global conformal transformation $z_1 \to z_2 = 1/z_1$ implies that $T(z_1)$ should transform as $T(z_2) = (dz_2/dz_1)^{-2}T(z_1) = z^4T(z_1)$. Therefore since we know $T(0)$ is finite (the regular part of the correlation function) this implies that $T(z_1)$ must scale as z_1^{-4} as $z_1 \to \infty$. Based on these arguments, this means that we can write down the correlation function $\langle T(z_1)T(z_2)\rangle$ as

$$\langle T(z_1)T(z_2)\rangle = \frac{c/2}{(z_1 - z_2)^4}, \tag{39}$$

where c is a constant fixed for a given system and $\bar{c}$ would correspond to the constant associated to the antiholomorphic version of eq. (39). This constant c is one of the most important object in CFT and is referred to as the central charge [1, 2, 3, 6, 21, 22]. The latter will be of utmost importance when dealing with minimal models. We can now use eq. (33) and explicitly derived an expression for $\delta_\beta T$ (as in eq. (38) we simply substitute ϕ by T under the assumption the latter is primary. This still holds but in a quantum theory, things tend to behave differently as we shall see). The fluctuation in $T(z)$ reads

$$\langle \delta_\beta T(z)\rangle = \frac{1}{2\pi i}\int_C dz\beta(z)\,\langle T(z)T(w)\rangle = \frac{1}{2\pi i}\int_C dz\beta(z)\frac{c/2}{(z - w)^4} = \frac{c}{12}\frac{d^3\beta(z)}{dz^3}, \tag{40}$$

where the Cauchy' s residue theorem was used. Hence it seems that now one has a symmetry breaking term on the far left hand side of eq. (40) which must be added to eq. (38). This is the conformal anomaly and the equivalent OPE is

$$T(z)T(w) \sim \frac{c/2}{(z - w)^4} + \frac{2T(w)}{(z - w)^2} + \frac{\partial T(w)}{(z - w)}, \tag{41}$$

[11] An important assumption is made here, we are actually treating $T(z)$ as a primary field which fine classically however in a quantum theory this transformation is altered due to normal ordering [1].

where a similar relation for the $T(\bar{z})T(\bar{w})$ exists. Specific models have specific non-vanishing c values fixed for each of these models [1, 3]. E.g: $c = 1$ for bosons and $c = 1/2$ for massless[12] fermions.

The emergence of the central charge, or the conformal anomaly in systems indicates symmetry breaking of the infinite-dimensional Lie algebra of conformal symmetry. The central charge also arises when introducing a macroscopic scale by working with a conformal field theory on a two dimensional curved surface [1, 20]. Hence the expectation value of the stress-energy tensor, $\langle T^{\mu}_{\mu}\rangle$ which is zero in flat space is not anymore in curved space but it equal to

$$\langle T^{\mu}_{\mu}(x)\rangle = \frac{c}{24\pi}R(x), \tag{42}$$

where $R(x)$ is the Ricci tensor. This is referred to as the trace anomaly in two dimensions and a complete derivation is found in Ref. [1] (Pg. 140, App. 5.A.) and further details in about the trace anomaly in $D > 2$ can be found in Ref. [20].

2.3 Simple case study: massless free fermion

We shall now consider the action for the massless Majorana fermion, χ [1]

$$S = \frac{1}{4\pi}\int d^2x\, \bar{\chi}\gamma^{\mu}\partial_{\mu}\chi, \tag{43}$$

where γ^{μ} are the Dirac matrices which closes under the Dirac algebra $\{\gamma^{\mu}, \gamma^{v}\} = 2\mathbb{I}\eta^{\mu\nu}$, where $\eta^{\mu\nu}=$ diag$(1, 1)$ in two dimensions. Using the following representation for the Gamma matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \gamma^0(\gamma^0\partial_0 + \gamma^1\partial_1) = 2\begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix}, \tag{44}$$

the action[13] (43) can be expressed in terms $\bar{\psi}$ and ψ where $\chi = (\psi, \bar{\psi})$ is the two component spinor such that

$$S = \frac{1}{4\pi}\int d^2x\, (\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi), \tag{45}$$

where $\bar{\partial} = \partial_{\bar{z}}$ and the equations of motion are $\partial\bar{\psi} = 0$ and $\bar{\partial}\psi = 0$. From the action above, we can then extract the OPE for the fermion field with itself which reads $\psi(z)\psi(w) \sim 1/(z - w)$ and similarly for the antiholomorphic sector. The sign of the latter is altered upon reversing the order of the fields as we are dealing with fermionic fields, this is also reflected later when using radial ordering in section (3.3). This result is achieved by calculating the propagator $\langle\chi_i(x)\chi_j(y)\rangle$ for $i, j = 1, 2$ upon using the kernel[14] $A_{ij}(\mathbf{x}, \mathbf{y}) = (1/4\pi)\delta(\mathbf{x} - \mathbf{y})(\gamma^0\gamma^{\mu})_{ij}\partial_{\mu}$ and the representations in (44). The two-point function is therefore $K_{ij}(\mathbf{x}, \mathbf{y}) = A^{-1}_{ij}(\mathbf{x}, \mathbf{y})$ A complete derivation can be found in Ref. [1] (Pg. 130) which is similar to the bosonic case [1] (Pg. 34).

We want the OPE of the stress-energy tensor with itself, in order to do so we recall the definition of the latter as

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\phi)}\phi^{\nu} - g^{\mu\nu}\mathcal{L}, \tag{46}$$

[12] We essentially treat massless cases since a theory with $m \neq 0$ at criticality breaks conformal invariance.

[13] We have already fixed the normalization which makes it easier to work with later on.

[14] A kernel of a path integral (i.e: a kernel of an integral transform) is the object which governs the time evolution of the system. It is also referred to as the propagator [24] of the system.

where ϕ is an arbitrary field with $\mathcal{L}$ corresponding to the integrand from (45) and then work with respect to ψ. The indices $\mu = 0, 1$ are replaced by $z, \bar{z}$ respectively such that normal ordered expression for the holomorphic and antiholomorphic stress-energy tensor read

$$T(z) = \frac{1}{2} : \psi(z)\partial\psi(z) : \qquad \bar{T}(\bar{z}) = \frac{1}{2} : \bar{\psi}(\bar{z})\bar{\partial}\bar{\psi}(\bar{z}) : \tag{47}$$

where the normal ordered product is $: \psi\partial\psi : (z) = \lim_{z\to w}(\psi(z)\partial\psi(w) - \langle\psi(z)\partial\psi(w)\rangle)$ and the normalization $T^{zz} = 2\pi T(z)$ (similarly for $\bar{T}(\bar{z})$) was used[15] [1]. After adequate successive Wick contractions and use of the OPE of $\psi(z)$ with itself, the OPE of $T(z)$ up to non-singular terms with itself then reads [1, 24]

$$
\begin{aligned}
T(z)T(w) &= \frac{1}{4} : \psi(z)\partial\psi(z) :: \psi(w)\partial\psi(w) : \\
&= \frac{1}{4}\left[\frac{\partial\psi(z)\partial\psi(w)}{(z-w)} - \frac{\partial\psi(z)\partial\psi(w)}{(z-w)^2} - \frac{\psi(z)\partial\psi(w)}{(z-w)^2} - 2\frac{\psi(z)\psi(w)}{(z-w)^3} + 2\frac{\psi(z)\psi(w)}{(z-w)^4} - \frac{\psi(z)\psi(w)}{(z-w)^4}\right] \\
&= \frac{1}{4}\left[\frac{(\partial\psi(w))^2}{(z-w)} - \frac{\partial\psi(w)\psi(w) - (z-w)\partial^2\psi(w)\psi(w)}{(z-w)^2} + \frac{\psi(w)\partial\psi(w) + (z-w)(\partial\psi(w))^2}{(z-w)^2}\right. \\
&\qquad \left. -2\frac{\psi^2(w)(z-w)\partial\psi(w)\psi(w) + 1/2(z-w)^2\partial^2\psi(w)\psi(w)}{(z-w)^3} + \frac{1}{(z-w)^4}\right] \\
&= \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2}{(z-w)^2}(1/2\psi(w)\partial\psi(w)) + \frac{1}{(z-w)}[\partial(1/2\psi(w)\partial\psi(w))],
\end{aligned}
$$

where we worked in the limit $z \to w$. From the second equality to the third one, we have used a trick to extend the terms. It can be noted that some terms in the third equality does cancel to give back the set of terms in the second line. The final form of OPE of $T(z)T(w)$ which is

$$T(z)T(w) = \frac{1/4}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial(T(w)) \tag{48}$$

where we have used the fact that $T(w) = (1/2\psi(w)\partial\psi(w))$ from (47). Upon comparing with definition in eq. (41), we note that $c = 1/2$ for fermionic field $\chi(z)$. The motivation of this example is based on the fact that the Ising model is identified as a massless free fermion at criticality. The case $c = 1/2$ will be shown later to corresponds to the minimal theory $\mathcal{M}(3, 4)$ whose critical exponents are found to correspond exactly to those of the 2D Ising model [2, 8, 9].

2.4 A discussion on the renormalization group and CFT

The modern stance on renormalisability was mostly motivated through the works of Wilson by providing an intuitive interpretation of scale dependence in quantum field theory. The essential idea is more about predictability rather than mathematical consistency of the physics at large scale [24]. In quantizing fields, an infinite number of degrees of freedom plague our quantum field theories such that naively performed quantum corrections diverge. Regularisation of momentum integrals is required which results in inserting an energy scale (which is completely arbitrary) by hand on which physical quantities will depend on hence preventing the divergence. This is essentially the bottom line of renormalisation. The typical renormalisation group (RG) illustrates a flow in the space of field theories generated by an alteration in the overall energy scale. For example, in statistical

[15] This explains why the factor of π from the action in (43) and (45) is not present in (47) anymore.

mechanical models we are generally presented with an action functional endowed with an initial cutoff Λ. Dependence on this initial cutoff is not important anymore when the correlation length, $1/\Lambda \to \infty$. In this regime, one can get rid of the dependence on the initial cutoff and include instead an arbitrary energy scale which would be associated to point of the renormalisation curve within the vicinity of an infrared fixed point. Actually $1/\Lambda \to \infty$ is characteristic of critical phenomena arising in a statistical mechanical system such that the removal of the initial cutoff is comparable to adjusting our system towards a fixed point. In this region near the infrared fixed point, the action functional would be characterised by $\{g^i\}$ [7, 24], where $\{g^i\}$ is the set of dimensionless coupling constants.

2.4.1 RG flow and the c-theorem

The RG flow is characterized by the Callan-Symanzik equation [24] which is derived in this subsection. We shall also demonstrate how the central charge, c of a two dimensional CFT appears as a result of the irreversibility of the RG flow [7] up to perturbation. We start with the variation of an action A under the infinitesimal variation $x^\mu = x^\mu + \epsilon^\mu(x)$

$$\delta_\epsilon A = \int d^2x \partial_\mu \epsilon_\nu(x) T^{\mu\nu}(x), \tag{49}$$

where we have A representing a string of fields $\phi_1(x_1)\phi_2(x_2)...\phi_n(x_N)$ such that the fields ϕ_i with $\phi_i \in \mathcal{A}$ where $\mathcal{A}$ is a vector space and assumed to be infinite dimensional. Hence we can write [6] (Ch. 3)

$$\sum_{i=1}^{N} \langle \phi_1(x_1)...\phi_{i-1}(x_{i-1})\delta_\epsilon\phi_i(x_i)\phi_{i+1}(x_{i+1})...\phi_N(x_N)\rangle =$$

$$\int d^2x \partial_\mu \epsilon_\nu(x) \langle T^{\mu\nu}(x)\phi_1(x_1)...\phi_N(x_N)\rangle \,, \tag{50}$$

where $\delta_\epsilon \phi$ will depend linearly on $\epsilon(x)$ as we' ve derived earlier. A closer look at the infinitesimal transformations [1, 6] we have $\epsilon_\mu(x) = \epsilon_\mu$, $\epsilon_\mu(x) = \omega_{\mu\nu}x^\nu$ where $\omega_{\mu\nu}$ is antisymmetric and lastly $\epsilon^\mu = (1/2)\, dt\, x^\mu$ corresponding to infinitesimal rotations, translations and scale transformations respectively. Considering D as the operator acting in $\mathcal{A}$ such that

$$\delta_\epsilon\phi(0) = dtD\phi(0), \qquad -i\delta_\epsilon\phi(x) = dt(x^\mu P_\mu + D)\phi(x), \tag{51}$$

where t is a renormalization group parameter and the operator δ_ϵ for translations is simply $i\epsilon^\mu P_\mu$ which has been used in the second equation in (51) leading to a reformulation of eq. (50)

$$\sum_{i=1}^{N} \left\langle \left(x_i^\mu \frac{d}{dx_i^\mu} + D_i \right) \phi_1(x_1)...\phi_N(x_N) \right\rangle = -\int d^2x \, \langle \Theta(x)\phi_1(x_1)...\phi_N(x_N)\rangle \,, \tag{52}$$

where we have substituted the second equation in (51) in the left hand side of (50) and Θ stands for the trace of $T_{\mu\nu}$. The integral on the left hand side of (52) actually diverges [6] such that we set D to contain the dependence on the cutoff parameters discussed earlier. Hence the description about the details of a field theory transforming under dilations is given by the renormalisation group.

We would like to be able to understand and explicitly write down the basic constituents of the renormalisation group associated to our original action functional A in (50). In order to proceed, one define first the following functional integral

$$\int \mathcal{D}\varphi(\phi_1(x_1)...\phi_N(x_N))\exp[-L(\varphi)], \qquad L(\varphi) = \int d^2x\mathcal{L}(\varphi(x), \partial_\mu\varphi(x)), \tag{53}$$

where $\mathcal{L}$ is an action density and $\varphi(x)$ represents a set of fields locally defined [6]. The set $\{g^a\}$ defined earlier is treated as coordinate system in A such that $\mathcal{L}$ would be a function of this general; infinite set, i.e: $\mathcal{L} = \mathcal{L}_g$. We define the local function, $\Psi_a(x) \in \mathcal{A}$ as

$$\Psi_a(x) := \frac{\partial}{\partial g^a}\mathcal{L}_g(\varphi(x)). \tag{54}$$

We proceed to differentiate (47) with respect to g^a to get

$$\frac{\partial}{\partial g^a}\langle(\phi_1(x_1)...\phi_N(x_N))\rangle =$$

$$\sum_{i=1}^{N}\langle\phi_1(x_1)...\phi_{i-1}(x_{i-1})B_a\phi_i(x_i)\phi_{i+1}(x_{i+1})...\phi_N(x_N)\rangle - \int d^2x\,\langle\Psi_a(x)(\phi_1(x_1)...\phi_N(x_N))\rangle\,, \tag{55}$$

where the $B_a\phi = \frac{\partial}{\partial g^a}A$. The second line in eq. (55) is obtained from the first line by simple application of the product rule and definition (54). The trace of the stress-energy tensor can be represented in terms of basis vectors as [6] which we understand by comparing (55) and (52)

$$\Theta(x) = \beta^a(g)\Psi_a(x), \tag{56}$$

where $\beta^a(g)$ are the components of the renormalisation group which are the well-known β-functions [6, 7]. The major part of this analysis of the renormalisation group is almost complete. By comparing eq. (52) and (55) and substituting the integral part of (52) in (55) we can finally obtain the Callan-Symanzik equation [6, 7, 24]

$$\sum_{i=1}^{N}\left\langle\left(x_i^\mu\frac{d}{dx_i^\mu} + \Gamma_i(g)\right)\phi_1(x_1)...\phi_N(x_N)\right\rangle = \sum_a \beta^a(g)\frac{\partial}{\partial g^a}\langle\phi_1(x_1)...\phi_N(x_N)\rangle\,, \tag{57}$$

where $\Gamma_i(g) = D - \beta^a B_a$ is a linear operator acting on the fields ϕ_i and is generally a matrix. From the Callan-Symanzik equation, we have that $dg^a = \beta^a(g)dt$ and the solution for $\beta^a(g)$ as mentioned by Zamolodchikov [7] is an indication of criticality (i.e: when $\beta^a(g_*) = 0$ at $g = g_*$, where g_* is identified as a fixed point).

We can now proceed to understand the c-theorem in two dimensions. As per the positivity condition [7, 24], the following set of two-points functions can be defined on $\mathbb{C}$,

$$\begin{aligned} 2z^4\langle T(z,\bar{z}), T(0,0)\rangle|_{z\bar{z}=1} &= C(g) \\ z^3\bar{z}\langle T(z,\bar{z}), \Theta(0,0)\rangle|_{z\bar{z}=1} &= H(g) \\ z^2\bar{z}^2\langle\Theta(z,\bar{z}), \Theta(0,0)\rangle|_{z\bar{z}=1} &= G(g), \end{aligned} \tag{58}$$

where we have the normalization $z\bar{z} = 1$ which is taken to be much larger than the ultraviolet (UV) cutoff for an arbitrary scale and $z\bar{z} = e^t$. Applying the condition that $\partial_\mu T^{\mu\nu} = 0$ followed by Callan-Symanzik equation (and the product rule) to the two point functions in (58) we end up with

$$\bar{z}\bar{\partial}(2z^4\langle T(z,\bar{z}), T(0,0)\rangle) + z\partial(z^3\bar{z}\langle T(z,\bar{z}), \Theta(0,0)\rangle) - 3\langle T(z,\bar{z}), T(0,0)\rangle = 0, \tag{59}$$

$$\bar{z}\bar{\partial}(z^3\bar{z}\langle T(z,\bar{z}),\Theta(0,0)\rangle) - z\partial\langle\Theta(z,\bar{z}),\Theta(0,0)\rangle)+$$
$$2\langle\Theta(z,\bar{z}),\Theta(0,0)\rangle) - \langle T(z,\bar{z}),T(0,0)\rangle = 0. \tag{60}$$

The above two equations can be also expressed as, using (58)

$$\dot{C} - \dot{H} + 3H = 0 \tag{61}$$

$$\dot{H} - H - \dot{G} + 2G = 0, \tag{62}$$

where the dot represents derivative with respect to t since we have defined t earlier to be equal to $\log(z\bar{z})$. We now multiply eq. (62) by 3 and add it to eq. (61) which leads to

$$\dot{c} = -12G, \tag{63}$$

where $c = 2F - 4H - 6G$ represents the famous c-function which is indeed monotonically decreasing because G is positive from positivity condition asserted earlier. Hence at fixed point, when $g = g_*$ we have that $\partial c(g_*)/\partial g^a = 0$ such that $\beta^a(g_*) = 0$ as previously inferred. Stationary points of c are therefore naturally fixed points and this relation is actually two fold and can be reached non-perturbatively from constraints imposed by the operator algebra and bootstrap methods. At fixed point, we know that the trace of the stress-energy tensor vanishes such that the last two equations in (58) equate to zero, hence we are left with

$$\langle T(z,\bar{z}),T(0,0)\rangle\,|_{g^a=g^a_*} = \frac{c(g^a_*)}{2z^4}, \tag{64}$$

where $c = c(g^a_*)$ represents the central charge of the Virasoro algebra in two dimensions. Hence classification of fixed point indeed lead to classification of CFTs (i.e: minimal models) which are characterized by the c value. This completes the proof of Zamolodchikov c-theorem.

3 The general structure of conformal field theory

In order to explore the consequences of conformal symmetries in two dimensions, we review in more details the procedure to construct a quantum theory of conformal fields. The theories we are working with are usually defined in the Euclidean space where we treat the holomorphic and anti-holomorphic sectors separately. This description of the holomorphic and anti-holomorphic left and right movers respectively in the complex plane make things computationally more convenient. Things can however be further simplified by conformally transforming the complex plane itself.

3.1 Radial quantization

In order to implement the technique of radial quantization we first define the mapping $z \to w = e^{2\pi z/L}$, where $z = z_1 + iz_2$ (or equivalently $\bar{z} = z_1 - iz_2$) which will map a cylinder (where our theories are usually defined) to the Riemann sphere, $\mathbb{S}^2 := \mathbb{C}\cup\infty$. The remote past on the cylinder extends to $-\infty$ while the remote future extends to $+\infty$ [1, 3], where this time coordinate is parametrised by z_1. The space along the circumference of the cylinder parametrised by z_2 where $z_2 \in [0, L]$. It can be noted that only one coordinate matters after the mapping. The points $z_1 = -\infty, +\infty$ is mapped to $z = 0, \infty$ respectively. A quantum conformal field theory is obtained by constructing operators that will lead to desired conformal mappings of the complex plane. This is illustrated by looking at a time translation $z \to z + \lambda$ which would simply be a dilation on the complex plane, $z \to e^{\lambda}z$. Hence the latter acts as a generator of the Hamiltonian and surfaces of constant radius defines the Hilbert space [3].

We proceed by assuming the existence of a vacuum state $|0\rangle$ which is used to construct a Hilbert space, $\mathcal{H}$ by repeated application of the creation operators and further assume that interacting fields just as free fields live in $\mathcal{H}$ but have different energy eigenstates. We can treat the interaction in such a way that it tends to zero in the limit $z_1 \to -\infty$ on the cylinder which corresponds to $z \to 0$ on the complex plane. Hence the asymptotic field can be expressed as

$$\phi_{in} = \lim_{z_1 \to -\infty} \phi(z_1, z_2). \tag{65}$$

In radial quantization, ϕ_{in} reduces to a single operator and defines an in-states by acting on $|0\rangle$

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle. \tag{66}$$

It is natural to look for a similar definition for the so called out-states $\langle\phi_{out}|$ i.e: when $z \to \infty$. Through the conformal symmetry, coordinates in the neighborhood of the point at ∞ to the coordinates of the region around the point $z = 0$ (origin) using the mapping[16] $z = 1/w$. For the out-state we therefore write

$$\langle\phi_{out}| = \lim_{w,\bar{w}\to 0} \phi(w,\bar{w}). \tag{67}$$

The above is then related to ϕ_{in} by the following transformation when $w \to g(w)$ where $g(w) = 1/w$, hence

$$\phi(w,\bar{w}) = \phi(g(w),\bar{g}(\bar{w}))(-w^{-2})^{h}(-\bar{w}^{-2})^{\bar{h}} = \phi\left(\frac{1}{w},\frac{1}{\bar{w}}\right)(-w^{-2})^{h}(-\bar{w}^{-2})^{\bar{h}}, \tag{68}$$

which follows from transformation properties of primary fields introduced previously. We can now used the above and substitute it in (67) such that

$$\langle\phi_{out}| = \lim_{w,\bar{w}\to 0}\langle 0|\phi(w,\bar{w}) = \lim_{z,\bar{z}\to 0}\langle 0|\left(\frac{1}{z},\frac{1}{\bar{z}}\right)\frac{1}{z^{2h}}\frac{1}{\bar{z}^{2\bar{h}}} = \lim_{z,\bar{z}\to 0}\langle 0|[\phi(\bar{z},z)]^{\dagger}, \tag{69}$$

where we have the adjoint representation of $\phi(z,\bar{z})$ in the last equality in (69) such that $\lim_{z,\bar{z}\to 0}\langle 0|[\phi(\bar{z},z)]^{\dagger} = [\lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0\rangle]^{\dagger}$ which is essentially $|\phi_{in}\rangle^{\dagger}$. Therefore $\langle\phi_{out}| = |\phi_{in}\rangle^{\dagger}$. This relation between in- and out-states will be important when constructing the operator algebra as they can be used to construct a Hermitian product as shown in Ref. [1] (Pg. 152).

3.2 Radial ordering

In quantum field theory (QFT) , the following correlation function defined as functional integrals in the Euclidean space is automatically time ordered (convergent) [1, 24],

$$\langle\phi_1 ... \phi_n\rangle = \lim_{\epsilon\to 0}\frac{\int[d\phi](\phi_1...\phi_n)\exp\{iS_{\epsilon}[\phi]\}}{\int[d\phi]\exp\{iS_{\epsilon}[\phi]\}}. \tag{70}$$

[16] We must note that the out-state is associated to Hermitian conjugation on conformal fields. This actually affects the Euclidean time $\tau = it$ leading to a reversal $\tau \to -\tau$ upon Hermitian conjugation. It is understood [3] that a reversal in parameters on the cylinder corresponds to an inversion on the complex plane when radially quantised (or $\mathbb{S}^2$ in our case) justifying our choice of mapping.

In radial quantization the same kind of ordering takes place and is referred to as radial ordering which is expressed as

$$\mathcal{R}\Psi_1(z)\Psi_2(w) \;=\; \begin{cases} \Psi_1(z)\Psi_2(w) \text{ if } |z| > |w| \\ \Psi_2(z)\Psi_1(w) \text{ if } |z| < |w|, \end{cases} \tag{71}$$

where the overall sign of the second term in (71) is negative if Ψ represents fermions. We shall relate to the operator product expansions. Both sides of an operator product expansion is to have an operator meaning according to (71). We now demonstrate how the OPEs can be related to commutation relations by first defining two holomorphic fields $\phi(z)$ and $\psi(w)$ by considering the following contour integral which we split into two fixed time contours going in opposite direction such that

$$\oint_w dz\mathcal{R}\phi(z)\psi(w) = \oint_{|z|>|w|} dz\phi(z)\psi(w) - \oint_{|z|<|w|} dz\psi(w)\phi(z) = [\Phi, \psi(w)], \tag{72}$$

where the integral on the left hand side is centered at w and $\Phi = \oint dz\phi(z)$ represents the contour integral of $\phi(z)$. We can start to see how the equal-time commutator $[\Phi, \psi(w)]$ is obtained from radially ordered product on left hand side of (72). This is true because for the two contours we have C_1 and C_2, the radii are initially set as being equal to $|w| + \epsilon$ and $|w| - \epsilon$ respectively and taking the limit $\epsilon \to 0$ justifies the last equality in (72) Essentially the commutator of two operators, Φ and Ψ is

$$[\Phi, \Psi] = \oint_0 dw \oint_w dz\phi(z)\psi(w), \tag{73}$$

where $\Psi = \oint dz\psi(w)$ and the integral over w is evaluated around the origin and the integral over z is evaluated around w.

3.3 Mode expansions

A central tool for studying conformal fields and conformal families is the mode expansion of the field $\phi(z, \bar{z})$ with conformal dimensions $(h, \bar{h})$ which we briefly discussed here by following closely the prescription of Ref. [1] (Pg. 153). The conformal field $\phi(z, \bar{z})$ can be mode expanded through its Laurent series (for both holomorphic and antiholomorphic sectors) as

$$\phi(z, \bar{z}) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \phi_{m,n}, \quad \phi_{m.n} = \left(\frac{1}{2\pi i}\right)^2 \oint dz z^{m+h-1} \oint d\bar{z}\bar{z}^{n+\bar{h}-1} \phi(z, \bar{z}). \tag{74}$$

By simply considering the definition of the adjoint of $\phi(z, \bar{z})$ used earlier in eq. (68) and use it on the left hand side of (74) and comparing terms upon simply taking the conjugate $\phi(z, \bar{z})^\dagger$ we end up with the following

$$\phi^\dagger_{m,n} = \phi_{-m,-n}, \tag{75}$$

which is the Hermitian conjugate of the expression on the left hand side of (74). For well-defined in- and out-states the action on $|0\rangle$ is

$$\phi_{m,n}|0\rangle = 0, \qquad (m > -h, n > \bar{h}). \tag{76}$$

3.4 The Virasoro algebra

In QFT, the generator of infinitesimal time translations is the Hamiltonian, $\hat{H}$ which is represented generally by the integral of T^{00}, i.e: the time component of the stress-energy tensor. This is a consequence of time translation symmetry in quantum mechanics. We have something analogous in CFT which was discussed in section (3.1) and it is the dilation operator, $\hat{D}$ on the complex plane which is seen as the Hamiltonian. This is a consequence of scale invariance. Hence we have

$$\hat{D} = \frac{1}{2\pi i}\left[\oint_0 dz\, zT(z) - \oint_0 d\bar{z}\, \bar{z}\bar{T}(\bar{z})\right] = L_0 + \bar{L}_0, \tag{77}$$

where the chosen contour is analytic such that it circles the origin and $L_0 + \bar{L}_0$ generates dilations. From (77) we can infer that L_n can $T(z)$ can be related using mode expansions (74) and similarly for the antiholomorphic components such that

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z), \quad T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \tag{78}$$

where we have used $h = 2$ in the second expression. A similar set of relations prevails for the antiholomorphic components, $\bar{L}_n$ and $\bar{T}(\bar{z})$ [1, 19]. In the same way the classical generators of local conformal transformations obey the Witt algebra derived in (22), the algebra resulting from commutation relations of generators L_n and L_m must close in the same way except for an extra term. We derive this algebra using eq. (73) and the OPE in eq. (41),

$$[L_n, L_m] = \left(\frac{1}{2\pi i}\right)^2 \oint_0 dw\, w^{m+1} \oint_w dz\, z^{n+1}\mathcal{R}T(w)T(z), \quad |z| > |w|$$

$$= \left(\frac{1}{2\pi i}\right)^2 \oint_0 dw\, w^{m+1} \oint_w dz\, z^{n+1}\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \text{reg.}\right],$$

where we can use Cauchy' s residue theorem in the limit $z \to w$ and evaluate the first contour integral over z;

$$= \left(\frac{1}{2\pi i}\right)\oint_0 dw\, w^{m+1}\left[\frac{1}{12}c(n+1)n(n-1)w^{n-2} + 2(n+1)w^n T(w) + w^{n+1}\partial T(w)\right]$$

$$= \left(\frac{1}{2\pi i}\right)\oint_0 dw\left[\frac{cn(n^2-1)/12}{w^{m+n}} + 2(n+1)w^{m+n+1}T(w) + w^{m+n+2}T(w)|_{w=0}\right.$$
$$\left. -(m+n+2)w^{m+n+1}T(w)\right],$$

the residue theorem is now used by evaluating the w integral for the first term, the third term is equal 0. We remark that the second and fourth terms above can written in terms of generators using (78),

$$= \frac{cn(n^2-1)}{12}\delta_{m+n,0} + 2(n+1)L_{m+n} - (m+n+2)L_{m+n}. \tag{79}$$

Hence the commutator of generators L_n and L_m is

$$[L_n, L_m] = \frac{cn(n^2-1)}{12}\delta_{m+n,0} + (n-m)L_{m+n}. \tag{80}$$

We note the second term on the left hand side of (80) is equivalent to eq. (22) for classical generators except that this algebra has an extension which depends on the central charge. The commutator $[L_n, L_m] = 0$ and a similar expression to (80) exists for $[\bar{L}_n, \bar{L}_m]$ and together with eq. (80), we have the Virasoro algebra, which we can denote as $\mathfrak{V}$ such that the conformal algebra is now given as $\mathfrak{V} \oplus \bar{\mathfrak{V}}$ where $\{L_{-1}, L_0, L_1\}+$ antiholomorphic sector are the generators of $\mathrm{SL}(2, \mathbb{C})$ in $\mathcal{H}$.

From eq. (76), it is understood that the vacuum $|0\rangle$ exhibits invariance under global conformal transformations. That is the latter is annihilated by L_n (and similarly by $\bar{L}_n$) for $n = \{-1, 0, 1\}$. Acting on this vacuum state $|0\rangle$ with primary fields creates eigenstates of the Hamiltonian H. This can be illustrated by evaluating the equal-time commutation relation as radially ordered products, substituting in the OPEs of $T(z)$ and $\phi(z, \bar{z})$ and simplifying using the residue theorem just like in (80), we end up with [1, 3]

$$[L_n, \phi(w, \bar{w})] = h(n+1)w^n\phi(w, \bar{w}) + w^{n+1}\partial\phi(w, \bar{w}), \tag{81}$$

where $n \geq -1$ and similarly for antiholomorphic counterpart. We therefore have

$$[L_n, \phi(0,0)] = 0 \quad \text{for} \quad n > 0, \tag{82}$$

$$L_0|h\rangle = h|h\rangle \quad L_n|h\rangle = 0 \quad \text{for} \quad n > 0, \tag{83}$$

where $|h\rangle = \phi(0)|0\rangle$ is an in-state (Hamiltonian eigenstate) created by acting on the vacuum with $\phi(0)$ having a conformal weight h. We have same relations when $h \to \bar{h}$ and $L_n \to \bar{L}_n$. The state in (83) is known as the highest weight state whose representations [9, 12] will be studied section (4.1). Excited states are then obtained by first looking at the commutation relation $[L_n, \phi_m]$ which is $[n(h-1) - m]\phi_{m+n}$ such that $[L_0, \phi_m] = -m\phi_m$. Hence for the eigenstate L_0, $\phi_{-m}(m > 0)$ act as a creating operator and $\phi_m, (m > 0)$ acts as annihilating operator. This also increases and decreases the conformal dimensions h. Hence excited states are obtained as

$$L_{-k_1}L_{-k_2}...L_{-k_n}|h\rangle \quad (1 \leq k_1 \leq k_2 \leq ... \leq k_n), \tag{84}$$

where the states above are known as descendants of primary state $|h\rangle$ and the eigenvalue is generally expressed as

$$h' = h + k_1 + k_2 + ... + k_n = h + N, \tag{85}$$

where N is the level of the descendant.

3.5 Conformal families

We explore a bit more the descendant fields, and their role in CFT. The remaining fields associated to the states, $L_{-n}|h\rangle$ in (85) are obtain by repeated application of the regular part of the OPE of $\phi(0)$ with the stress-energy tensor. The descendant field is therefore defined as [1]

$$\phi^{-n}(w) \equiv (L_{-n}\phi)(w) = \frac{1}{2\pi i}\oint_w dz(z-w)^{1-n}T(z)\phi(w). \tag{86}$$

From the OPE of $T(z)$ with $\phi(w)$, we can identify the following relations

$$\phi^{(0)}(w) = h\phi(w), \qquad \phi^{(-1)}(w) = \partial\phi(w). \tag{87}$$

The correlation functions of the descendants can be understood by understanding the properties of the primary field. In order to do so, we consider the correlator

$$\langle L_{-n}\phi)(w)X\rangle\,, \tag{88}$$

where $X = \phi_1(w_1)...\phi_N(w_N)$ represents a string of primary fields with holomorphic conformal dimensions h_i. Using eq. (86) we reverse the contour around w_i by denoting the boundary as $\{w_i\}$ and evaluate using the residue theorem. From the OPE in eq. (37) and (88), we have

$$\begin{aligned}\left\langle \phi^{-n}(w)X\right\rangle &= \frac{1}{2\pi i}\oint_w dz\,(z-w)^{1-n}\,\langle T(z)\phi(w)X\rangle \\ = -\frac{1}{2\pi i}\oint_{\{w_i\}} dz &\sum_i \left[\frac{(z-w)^{1-n}}{z-w_i}\partial_{w_i} + \frac{(z-w)^{1-n}}{(z-w_i)^2}h_i\right]\langle\phi(w)X\rangle \\ &= \sum_i\left[-\frac{\partial_{w_i}}{(w-w_i)^{n-1}} + \frac{(n-1)}{(w_i-w)^n}h_i\right]\langle\phi(w)X\rangle\,,\end{aligned} \tag{89}$$

where the residue theorem was used to get the last equation. The expression in the square brackets is defined as

$$\mathcal{L}_n := \sum_i\left[-\frac{\partial_{w_i}}{(w-w_i)^{n-1}} + \frac{(n-1)}{(w_i-w)^n}h_i\right],\quad n\geq 1. \tag{90}$$

Eq. (90) is a differential operator which is used to obtain the derivative of the primary field, i.e: the descendants. We can generalize (80) by recursively applying this differential operator which leads to

$$\left\langle \phi^{(-k_1,\ldots,-k_n)(w)X}\right\rangle = \mathcal{L}_{-k_1}...\mathcal{L}_{-k_n}\,\langle\phi(w)X\rangle\,. \tag{91}$$

Hence a conformal family $[\phi]$ is essentially a set of primary field (the ancestor field) ϕ and all of its descendants with conformal dimensions h_i and also comprising of the whole antiholomorphic sector.

3.5.1 The operator algebra

As seen above, we can argue that all the data concerning the conformal field theory is enclosed in the correlation function which forms the principle entity in field theory. Conformal invariance imposes strong constraints on the 2- and 3- point functions and allows us to derive explicitly the conformal data (i.e: spins and scaling dimensions). However the information derived from conformal symmetry is limited to the 3-point and still then some data needs to be inserted by hand [1, 22]. In order to be able to fully solve our theory, we invoke the operator algebra which provides us with more constraints on top of those from conformal invariance. The algebra of local fields was devised and extensively researched by the authors from Ref. [2, 13, 14, 21] in order to calculate critical exponents exactly.

We will now look at the OPE in a slightly more formal way than in the previous sections. From eq. (26) we know that 2-point functions vanishes if the conformal dimensions differs. In the complex plane, the former reads

$$\langle\phi_m(z,\bar{z})\phi_n(w,\bar{w})\rangle = C_{mn}\frac{1}{(z-w)^{2h}}\frac{1}{(\bar{z}-\bar{w})^{2\bar{h}}}, \tag{92}$$

where C_{mn} is a symmetric coefficient. The OPE is generally defined [3] as $\Psi(x)\Phi(y) \sim \sum_i C_i(x-y)O_i(y)$, where O_i the product is expanded in terms of a basis of local operators O_i with coefficients C_i. The latter depends on the coordinates such that $C_i \sim |x-y|^{d_{O_i}-d_\Psi-d_\Phi}$ from simple dimensional analysis. The OPE for (86) then reads

$$\phi_m(z,\bar{z})\phi_n(w,\bar{w}) \sim \sum_\gamma C_{mn\gamma}(z-w)^{h_\gamma-h_m-h_n}(\bar{z}-\bar{w})^{\bar{h}_\gamma-\bar{h}_m-\bar{h}_n}\phi_\gamma(w,\bar{w}), \tag{93}$$

where $C_{mn\gamma}$ is symmetry under permutation of indices. The issue that is presented here is the difficulty in writing OPE for the descendants in terms of primary fields. Orthogonality of primary fields [1, 2, 3] implies orthogonality of the descendants, hence from scale invariance we express (93) in terms of secondary (descendant) fields via the OPE the operator algebra reads

$$\phi_m(z,\bar{z})\phi_n(0,0) \sim \sum_p \sum_{\{k,\bar{k}\}} C_{mn}^{p\{k,\bar{k}\}}(z)^{h_p-h_m-h_n+K}(\bar{z})^{\bar{h}_p-\bar{h}_m-\bar{h}_n+\bar{K}}\phi_p^{\{k,\bar{k}\}}(0,0), \tag{94}$$

where the $C_{mn}^{p\{k,\bar{k}\}}$ is the coefficient of the operator algebra[17] and $\phi_p^{\{k,\bar{k}\}}(0,0)$ represents the set of descendants fields. Again dimensional analysis is useful in determining the respective conformal dimensions above. Since properties of descendants follows from properties of primary fields, we can argue that the coefficient in (94) takes the following form

$$C_{mn}^{p\{k,\bar{k}\}} = C_{mn}^{p}\beta_{mn}^{p\{k\}}\bar{\beta}_{mn}^{p\{\bar{k}\}}, \tag{95}$$

where the coefficients $\beta_{mn}^{p\{k\}}$ and $\bar{\beta}_{mn}^{p\{\bar{k}\}}$ are determined in terms of the central charge by requiring that both sides the operator algebra obey conformal symmetry. A complete example can be found in Ref. [1] (Pg. 181-182) however it can be noted that dealing with cases when the descendant level get slightly larger becomes very tedious even though the theory is solved exactly. The coefficients for higher point functions have to be determined via some other method and the conformal bootstrap, introduced in the following subsection, provides an adequate formalism in order to do so.

3.5.2 Conformal blocks, duality and the bootstrap

The bootstrap formalism introduced in this section is based on Refs. [26, 27, 29][18]. We will be denoting scalars by $\Psi_i(x)$ in this section. The bootstrapping procedure is started by first recalling the definition of the OPE in section (3.5.1) which can be expressed[19] as

$$\Psi_i(x)\Psi_j(0) = \sum_k C_{ijk}(1+\partial+...)\Psi_k(0) = \sum_k C_{ijk}\mathcal{F}(x,\partial)\Psi_k(0), \tag{96}$$

[17] This is the coefficient that Zamolodchikov reached in deriving the c-theorem [7]. The latter was obtained via perturbative methods where $\epsilon(x) << 1$.

[18] This is the bootstrapping procedure developed by R. Rattazi et al. in Ref. [29] providing a novel way to construct conformal blocks, i.e: via a geometrical approach. This method so far has been successful as noted by the accurate bounds obtained in [27, 29] and can be applied to fermions. This is because such framework can provide a window for explaining the data presented in [28], where minimal models are seemingly realized at the boundary of a topological superconductor. Due to time constraints it was not possible to go through the details however the plan for future research shall begin from the framework discussed above but applied to fermions.

[19] This is essentially writing OPEs like (37) in a more compact way and generalizing for the rest of the less singular terms. We also note that (96) has an equal sign instead of the usual $\sim$ symbol. There are two reasons for that: (i) Eq. (96) is an exact result that can be employed in path integrals [26]. (ii) The difference between the axiomatic formulation of a product in CFT and the OPE in (96) is about convergence which is crucial for numerical analysis [22, 26] but analytically we can work with both.

where $\mathcal{F}(x, \partial) = (1 + \partial + ...)$ represents the descendants contribution which we know from (95) [1](Pg. 181) is fixed by conformal invariance and C_{ijk} is the OPE coefficient. The question is how to determine $D(x, \partial)$ for 2-, 3- and n-point functions? One way to proceed is to write down the 3-point function and reduce to a 2-point function

$$\langle \Psi_1(x_1)\Psi_2(x_2)\Psi_3(x_3)\rangle = \sum_k C_{12k}\mathcal{F}(x_{12}, \partial_2) \langle \Psi_2(x_2)\Psi_3(x_3)\rangle\,, \tag{97}$$

From eq. (26) and (27), we work out both the l.h.s and r.h.s of (97)

$$\frac{C_{123}}{x_{12}^{h_1+h_2-h_3} x_{23}^{h_2+h_3-h_1} x_{13}^{h_3+h_1-h_2}} = C_{123}\mathcal{F}(x_{12}, \partial_2) \langle \Psi_2(x_2)\Psi_3(x_3)\rangle\,, \tag{98}$$

such that

$$\frac{1}{x_{12}^{h_1+h_2-h_3} x_{23}^{h_2+h_3-h_1} x_{13}^{h_3+h_1-h_2}} = \mathcal{F}(x_{12}, \partial_2)\left(\frac{1}{x_{23}^{2\Delta_3}}\right), \tag{99}$$

where $\mathcal{F}(x_{12}, \partial_2)$ can be fully solved by relating a 2- and a 3-point function using conformal invariance. Information concerning the CFT in question is therefore reached faster than the method described in [1] (Pg. 181). These results seem promising so we proceed along the same line for the 4-point function,

$$\langle \Psi_1(x_1)\Psi_2(x_2)\Psi_3(x_3)\Psi_4(x_4)\rangle = \sum_k C_{12}^k C_{34}^k \mathcal{F}(x_{12}, \partial_2, x_{34}, \partial_4) \langle \Psi_2(x_2)\Psi_4(x_4)\rangle\,, \tag{100}$$

where we have simply used the fact that $\Psi_1(x_1)\Psi_2(x_2) = \sum_k C_{12}^k \mathcal{F}(x_{12}, \partial_2)\Psi_k(x_2)$ and $\Psi_3(x_3)\Psi_4(x_4) = \sum_k C_{34}^k \mathcal{F}(x_{34}, \partial_4)\Psi_k(x_4)$. The l.h.s of (94) is therefore equal to

$$\sum_k C_{12}^k C_{34}^k \mathcal{F}(x_{12}, \partial_2, x_{34}, \partial_4)\left(\frac{1}{x_{24}^{2\Delta_k}}\right), \tag{101}$$

such that for a 4-point function we end up with two $\sum \mathcal{F}$ acting on a 2-point function. From conformal symmetry, the form of the 4-point function is

$$\langle \Psi_1(x_1)\Psi_2(x_2)\Psi_3(x_3)\Psi_4(x_4)\rangle = \frac{g(u, v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}, \tag{102}$$

where u and v are respectively defined[20] as $(x_{12}^2 x_{34}^2)/(x_{13}^2 x_{24}^2)$ and $(x_{23}^2 x_{14}^2)/(x_{13}^2 x_{24}^2)$. The latter are conformally invariant therefore does not provide enough constraint to fix the 4-point function. By trivially inserting the prefactor $1/(x_{12}^{2\Delta} x_{34}^{2\Delta})$ in (101) and comparing with (102), the function $g(u, v)$ reads

$$g(u, v) = \sum_k C_{12}^k C_{34}^k \mathcal{F}(x_{12}, \partial_2, x_{34}, \partial_4)\left(\frac{1}{x_{24}^{2\Delta_k}}\right) = \sum_k C_{12}^k C_{34}^k g_{\Delta_k, s_k}(u, v), \tag{103}$$

where $g_{\Delta_k, l_k} = \mathcal{F}(x_{12}, \partial_2, x_{34}, \partial_4)(1/x_{24}^{2\Delta_k})$ represents the conformal data (i.e: scaling dimensions Δ_i and spins s_i) and The functions defined above in (103) are called the conformal blocks[21] [1, 2, 26]. Only the holomorphic sector was considered in deriving (97)

[20] u and v are also expressed as $z\bar{z}$ and $(1 - z)(1 - \bar{z})$ respectively as this provides a way for explicit solution of $g(u, v)$. A complete derivation of the latter can be found in Ref. [26] (Pg. 45) using a second order differential is solved leading to some hypergeometric function analysis and also via using series expansion in terms of radial coordinates.

[21] This expressions corresponds to eq. (6.188) in Ref. [1] (Pg. 185).

for simplicity but the antiholomorphic components bear a similar structure to the former.

These expressions are symmetric under contraction of the indices $(12;34) \leftrightarrow (14;23)$ such that one can write

$$\sum_k C_{12}^k C_{34}^k g_{\Delta_k,s_k}(u,v) = \sum_k C_{14}^k C_{23}^k g_{\Delta_k,s_k}(v,u), \tag{104}$$

where the exchange $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$ simply implies the exchange $u \leftrightarrow v$ when looking at the definitions for u and v given above under (102). This duality between the conformal blocks under contraction of the indices is referred to as crossing symmetry [1, 2, 26] and as one can note it imposes a very strong (non-perturbative) constraint for determining the conformal data; Δ_i and coefficients C_{mn}^k. Eq. (104) is central is providing us with information required to solve our theory and much progress was made in solving the former in $d \geq 3$ [23].

3.5.3 The four point function

The critical exponents of the four-point function of the Ising model can be calculated using (102) and (103). The four-point function will be the four spin correlator (since the Ising model is identified by the spin operator σ and its energy operator ε), hence

$$\langle \sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle \sim \frac{1}{|z_{12}z_{34}|^{1/4}}\left[(C_{\sigma\sigma\mathbb{I}})^2 + (C_{\sigma\sigma\varepsilon})^2 \frac{1}{z_{24}^{2\Delta}} + \ldots\right] \sum_k \mathcal{F}(x_k, \partial_k), \tag{105}$$

where $\mathcal{F}(x_k, \partial_k)$ as mentioned above and in Ref. [2, 3] can be simplified since it is, in two dimensions, a representation of the anharmonic ratios (28). In Ref. [2] (App. E), the four-point is shown to satisfy a second order differential equation whose solution leads to hypergeometric function which is reduced into an elementary function. The latter can then be Taylor expanded. By comparing these two methods we can find explicit values of structure constant C_{ijk} in (99). From the second method, have

$$\langle \sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle \sim \frac{1}{2}\left|\frac{z_{13}z_{24}}{z_{12}z_{34}z_{14}z_{23}}\right|^{1/4} \sum_{a,b=1}^{2} f_a(x) f_b(\bar{x}), \tag{106}$$

where $x = (z_{12}z_{34})/(z_{13}z_{24})$ represents the cross-ratios and $\sum_{a,b=1}^{2} f_a(x)f_b(\bar{x}) = (|1 + \sqrt{1-x}| + |1 - \sqrt{1-x}|)$ [3] (Pg. 65). Hence the latter can be Taylor expanded in the regime $x \to 0$ and by comparing (106) with the definition of $f(x)$ and (105), the structure constants read

$$(C_{\sigma\sigma\mathbb{I}})^2 = 1 \quad \text{and} \quad (C_{\sigma\sigma\varepsilon})^2 = \frac{1}{4}. \tag{107}$$

These results corresponds exactly to the critical exponents of the Ising model obtained using operator algebra methods [2, 13, 14, 21] compared to RG flow methods [1](Ch. 3), [31] and [6] (Ch. 5) where critical exponents are derived perturbatively.

4 Representations of the Virasoro algebra

It is now unequivocal that theories admitting conformal invariance are characterized by the value of the conformal charge c and the set of holomorphic (and antiholomorphic) conformal dimensions h (and $\bar{h}$). Moreover the critical exponents which were briefly discussed in the previous chapter described the universality class in statistical physics.

One can therefore ask if it is possible to actually find a given value of c that will determine all the critical exponents of our theory. The answer is yes and the values of c that we find correspond to sets of conformally invariant minimal theories admitting universality class. This is exemplified by constructing and studying the irreducible representations of the Virasoro algebra, $\mathfrak{V} \oplus \bar{\mathfrak{V}}$.

4.1 Highest weight representations

The representation theory of the Virasoro algebra is very similar to the algebra of angular momentum, $\mathfrak{su}(2)$. We therefore recall the algebraic theory of angular momentum from quantum mechanics [1, 24] as a starting point. The representation space of $\mathfrak{su}(2)$ is spanned by states labeled by the eigenvalue J_z which is one of the generators. The two others generators are annihilation J_+ and creation J_- operators. The highest weight state $|j\rangle$ is defined as the one with eigenvalue J_z, which is acted up by J_- to obtained the other eigenstates. For the eigenstate $|j\rangle$, we have

$$J_z|j\rangle = j|j\rangle \quad \text{and} \quad J_+|j\rangle = 0, \tag{108}$$

where the algebra of the generators closes as: $[J_0, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_z$. The generators are also Hermitian, i.e: $J_i^\dagger = J_i$. Using the definition for a state $|n\rangle = (J_-)^{j-m}|j\rangle$ and we know that $|n-1\rangle = J_+|n\rangle$ such that $\langle n-1|n-1\rangle = J_- J_+ \langle n|n\rangle$. Using the definitions (108) and the fact that j^2 is the Casimir operator (commutes with all generators close to identity), we end up with the following relation

$$\langle n-1|n-1\rangle = [j(j+1) - n(n-1)] \langle n|n\rangle \,, \tag{109}$$

where the dimension of this representation is $2j+1$ which is finite and unitary. However it can be noted that when n is reduced below $-j$ leads to negative norm states unless j is an integer or half-integer [24]. Hence these negative norm states are not accounted by the first $2j+1$ states. The former then constitute null states (or singular vectors) which will be important later on when discussing irreducible representations of the Virasoro algebra.

In the case of the Virasoro algebra (as introduced in subsection (3.4)), representations are obtained by taking the tensor products of the irreducible left (holomorphic) and right (antiholomorphic) sectors. For the Virasoro algebra $\mathfrak{V}$ (for the holomorphic sector) for simplicity, the highest weight state is generated by L_0[22] which diagonalises the representation space forming a Verma module [30]. A Verma module is representation of $\mathfrak{V}$ which consists of the Hilbert subspace spanned by the in-state (or asymptotic state) $|h\rangle$ and its descendants which closes under the action of the Virasoro generators [1] (Pg. 158 for definition). The highest weight state is expressed as $|h\rangle$ which is defined in eq. (83) as an asymptotic state created by acting on the conformally invariant vacuum $|0\rangle$. The commutation relation $[L_0, L_m] = -mL_m$ allows us to define the creation operator as $L_{-m}(m>0)$ and the annihilation operator as $L_m(m>0)$ and descendants states are created in the same way as described that is by acting repeatedly on the $|h\rangle$ by $L_{-k_1}...L_{-k_n}(1 \leq k_1... \leq k_n)$. The first few states of the Verma modules are given in Ref. [1] (Pg. 202).

[22]It is reasonable that the highest weight representation is L_0 since $L_0 + \bar{L}_0$ encountered in section (3.4) is the Hamiltonian (in radial quantization) which we know is bounded from below [1, 2].

4.2 Verma modules and singular vectors

The Verma modules form a unitary infinite dimensional representation space[23] of the Virasoro algebra. Unitarity of the modules is reflected by the Hermiticity condition $L_m^\dagger = L_{-m}$ which implies that the levels N given in (85) are orthogonal. This motivates the presence of a positive definite inner product which can be expressed as

$$\langle h|L_{k_m}...L_{k_1}L_{-l_1}...L_{-l_n}|h\rangle\,, \tag{110}$$

where the latter is evaluated using the Virasoro algebra. One can consider the product as an example; $\langle h|L_kL_{-k}|h\rangle = \langle h|[L_k, L_{-k}]|h\rangle = c(k^3-k)/12 + 2kh$. Hence the products are evaluated by passing the L_{k_j} over the L_{k_i} [1]. This example is an important example inferring that an essential condition for unitarity is $h \geq 0$ and also that $c \geq 0$ [9, 12]. One might wonder about structure of the Hilbert space if the bound on c is altered and the implications on unitarity in general. This require a slightly more sophisticated approach[24] which will covered in sections (4.3) and (4.4). Hence for the case presented above, the Hilbert space $\mathcal{H}$ is generally expressed as

$$\mathcal{H} = \bigoplus_{h,\bar{h}} V(c,\bar{h}) \otimes \bar{V}(c,\bar{h}), \tag{111}$$

that is a direct sum of the tensor of the holomorphic and antiholomorphic Verma modules denoted as $V(c,h)$ generated by the sets $\{L_n\}$ and $\bar{V}(c,\bar{h})$ generated by $\{L_{-n}\}$ respectively for a given value of c.

It is understood that the representations of the Virasoro algebra we found and discussed above are irreducible [2, 8, 9] otherwise we would not have been able to write down eq. (105). However it is possible that the representations of $\mathfrak{V}$ are reducible. In that case, just as the Verma modules $V(c,h)$ and its antiholomorphic counterpart is generated by acting on $|h\rangle$, the submodule present is generated by the highest weight state which is now denoted[25] by $|\chi\rangle$ such that $L_n|\chi\rangle = 0\,(n>0)$ and $L_{-r_n}...L_{-r_1}|\chi\rangle = 0\,(1 \leq r_1 \leq ... \leq r_n)$. Hence a state that is destroyed by all $L_n\,(n>0)$ is known as a singular vector (or null state) where the latter possesses its own Verma module, V_χ leading to a discussion similar to that of the last paragraph above. Through the operator-state correspondence [25], the singular vector $|\chi\rangle$ is related to the null field field $\chi(z)$ (and similarly for the antiholomorphic null field)[26] If the Verma module contain numerous null states, the irreducible representations of the Virasoro algebra are given by taking the quotient of $V(c,h)$ with respect to the set of singular vectors $\{\chi_i\}$ $(i=1...r)$ such that

$$M(c,h) := V(c,h)/\{\chi_i\}, \quad i = 1...r, \tag{112}$$

$M(c,h)$ represents the actual representation involved in the construction of minimal models.

[23] Space is now infinite dimensional unlike representation space of $\mathfrak{su}(2)$ which is finite.

[24] The more formal references can be consulted; Ref. [9] (Th. 1) provides a brief discussion on unitarity when $c \leq 1$. Ref. [12] for elaboration of the proof in [9] and finally Ref. [10] which tackles the subject by implementing the affine Kac-Moody algebras for finding the discrete set of non reducible highest weight representation of $\mathfrak{V}$.

[25] We follow the notation given in the short section of Ref. [1] (Pg. 204) as discussions on singular vectors prevail mostly in mathematical literature [30] and escalate very quickly drifting away from applications in CFT.

[26] $\chi(z)$ is primary in the Verma submodule but it is secondary since it is a descendant $\phi(z)$ from the Verma module (representation of the Virasoro algebra).

4.3 The Kac determinant

We want to investigate how strong is the bound on unitarity imposed by the Hermicity condition. The fact that the norm, $||L_{-k}|h\rangle||^2 = \langle h|L_k L_{-k}|h\rangle \geq 0$ if $h \geq 0, c \geq 0$ already seems to impose interesting constraints on the parameters h and c. When $c < 0$, representations are non-unitary and does not relate to known minimal theories. In this subsection we will construct the representations of $\mathfrak{V}$ up to level 3. The data that we will be collecting by applying the definition of the norm can be inserted in a matrix denoted as $M^{(k)}(c,h)$, where k is the level of the representation. At level 0, no relevant information prevails thus from normalization condition, $\langle h|h\rangle = 1$ we have[27] $M^0 = 1$. We list the representations for the first four levels and a complete construction of $M^{(l)}(c,h)$ up to $l = 3$ in table. (1) below. At level $l = 1$, we retrieve the condition $h > 0$ as imposed by positive definiteness of the norm seen above. When $l = 2$ we compute the determinant of $M^2(c,h)$ which is

$$\det M^2(c,h) = 32h^3 - 20h^2 + 4h^2c + 2hc = 32(h - h_{1,1})\left(h^2 - \frac{5}{8}h^2c + \frac{1}{8}hc + \frac{1}{16}c\right), \quad (113)$$

such that the determinant is given by the product of two eigenvalues of $M^{(2)}$. The expression in the big brackets can be easily factorized as $(h - h_{1,2})(h - _{2,1})$ where the solutions for (113) then reads: $h_{1,1} = 0, h_{1,2} = (1/16)[5 - c - \sqrt{(1-c)(25-c)}]$ and $h_{2,1} = (1/16)[5 - c + \sqrt{(1-c)(25-c)}]$.

From the above solutions, it can be argued that the interesting cases to look at are: (i) When one of the eigenvalues of $M^l(c,h)$ is negative (or when l is such that we have an odd number of negative eigenvalues) then $\det M^l(c,h) < 0$. This is forbidden and negative norm state will immediately imply nonunitary theories which are discarded. (ii) When $\det M^l(c,h) = 0$, this now implies that one of the eigenvalues of $M^l(c,h)$ is zero suggesting the presence of null states at $l = 1$ (i.e: $L_{-1}|0\rangle = 0$ will coincide with the line $h = 0$ on the h-c curve [1]). (iii) Finally if $\det M^l(c,h) > 0$, then there could be now an even number of null states. Null states, which we know are defined as $L_{-k_1}...L_{k_n}|h+n\rangle = 0$, suggest for some h-value at level k, the determinant will have a zero of order $p(l-k)$. Hence (107) is proportional to $(h - h_{1,1})^{p(1)}$. For a linear combination of states with zero norm, this definition of the determinant can be generalized to account for all the zero eigenvalues occurring at level $l = pq$, where p, q are integers. Hence one have

$$\det M^l(c,h) = \alpha_l \prod_{rs \leq l} [h - h_{r,s}(c)]^{p(l-rs)}, \quad (114)$$

where $\alpha_l > 0$ is a constant and $p(l - rs)$ is the number of partitions. The expression in (115) is known as the Kac determinant and was explicitly proven in Ref. [30] (Th. 3.1, Pg 124). The roots for $h = h_{r,s}$ and c can be expressed under reparametrization [1] (Ch. 8) as

$$c = 1 - \frac{6}{m(m+1)} \quad \text{and} \quad h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \quad (115)$$

where the allowed values for m are $3, 4, 5, ...$ and the restrictions on r and s are $1 \leq r \leq m-1$ and $1 \leq s \leq r$. A geometrical interpretation of all this data is given in Figure. (1) on Pg. 28 but one can start to see that from (108) that the first null state in the reducible Virasoro Verma module $V(c, h_{r,s})$ occur at level $l = rs$ since the determinant vanishes.

[27] One can consult Ref. [1] (Pg. 206) for notation details. Note that $M_{ab} = \langle a|b\rangle = 1$ iff $a = b$ i.e: level 0 and $M^\dagger = M$. M is known as the Gram matrix [1].

Furthermore the points on the curves (all of them actually but there are exception which will be discussed in the next subsection) are associated to the reducible Verma modules, $V(c, h_{r,s})$ and the region $c < 1$ is nonunitary which implies that the regions enclosed by the vanishing curves enclosed are all nonunitary as well [2, 19].

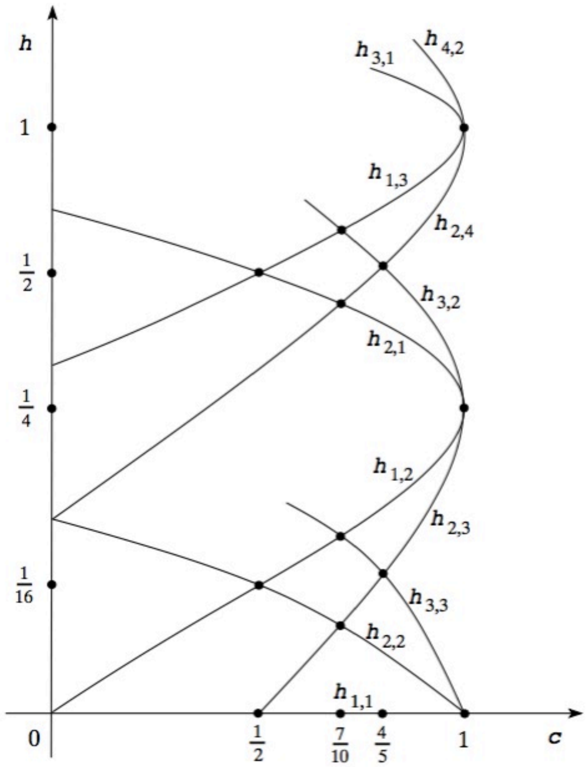

Figure 1: The vanishing curves for the few values of $h = h_{r,s}$ on h-c plane. Image from Ref. [3]

4.4 Unitary representations

4.4.1 $c \geq 1$

We have made an interesting claim above that $c < 1$ represents nonunitary theories. A sketch of this proof can be found in Ref. [9] and [12]. However we first convince ourselves that the region $c \geq 1, h > 0$ is unitary. One can recall that before a less strict bound on c, where $c \geq 0$ was shown to exist from positive definiteness condition. We now extend this argument to $c \geq 1$ and follow the steps of Ref. [1]. From the solutions of (113), $h_{r,s}(c)$ is not real for $1 < c < 25$ leading to an imaginary part (or even ≤ 0). For $c \geq 25$ we are free to choose the constraint $-1 < m < 0$ such that all $h_{r,s}(m) \leq 0$ from eq. (115) (Note one goes from $h_{r,s}(c) \to h_{r,s}(m)$ because we are working with two different reparametrization[28] of $h_{r,s}$ [1]). This means that $\det M^l(c,h)|_{c \geq 25} > 0$ indicating all positive eigenvalues, and for large h, $\det M^l(c,h) \approx \alpha h$ from eq. (114), hence this is

[28] $h_{r,s}(c)$ can expressed differently, one way is $h_{r,s}(c) = h_0 + (1/4)(r\alpha_+ + s\alpha_-)^2$ where $h_0 = (1/24)(c-1)$ and $\alpha_\pm = (\sqrt{1-c} \pm \sqrt{25-c})/\sqrt{24}$ such that upon substituting directly h_0 and $\alpha_\pm$ in $h_{r,s}(c)$, one end up with

$$h_{r,s}(c) = \frac{1-c}{96}\left\{\left[(r+s) + (r-s)\sqrt{\frac{25-c}{1-c}}\right]^2 - 4\right\}.$$

Level l	Representations of $V(c,h)$	$M^{(l)}(c,h)$
0	$\lvert h\rangle$	$M^{(0)}(c,h) = 1$
1	$L_{-1}\lvert h\rangle$	$M^{(1)}(c,h) = 2h$
2	$L_{-1}^2\lvert h\rangle, L_{-2}\lvert h\rangle$	$M^{(2)}(c,h) = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h+c/2 \end{pmatrix}$
3	$L_{-1}^3\lvert h\rangle, L_{-1}L_{-2}\lvert h\rangle, L_{-3}\lvert h\rangle$	$M^{(3)}(c,h) = \begin{pmatrix} 24h(1+h)(1+2h) & 12h(1+3h) & 24h \\ 12h(1+3h) & h(8+c+8h) & 10h \\ 24h & 10h & 2c+6h \end{pmatrix}$

Table 1: Evaluation of the Gram matrix $M^{(l)}(c,h)$ for the first (lowest) states of the Virasoro Verma representation, $V(c,h)$

all positive and non-zero in the region $c > 1, h > 0$ leading to unitary theories in the domain in question. The case then $c = 1$ acts as a threshold between nonunitarity and unitarity and does not provide further information such that points lying on line $c = 1$ will also be unitary.

4.4.2 $c < 1$

The sector when $c < 1$ is not easy to assess, it was claimed and proven by the authors in Ref. [9] that even if the norm of the states lying in the region $0 < c < 1$ are negative thus leading to nonunitarity, there exists a discrete set of points that fits the representation in eq. (115) which exhibits unitarity, Such proof require coset construction which is discussed in Ref. [1] (Ch. 18) and goes beyond this work. However it can be easily understood from the basic principles employed above. Let' s consider the matrix $M^l(c,h)$ explicitly for each level ($l = 1, 2, 3, \ldots$). We know that these points are all reducible so we can find a subset for each l, say g_l where $g_l \in M^l(c,h)$ at that level l for $c < 1$ such that eigenvalues of $M^l(c,h)$ as expected are negative. According to authors of Ref. [9], this subset and all other similar subsets associated at all the other levels l can be all combined (i.e: taking the unions of the sets, g_l) and remove from these sets from the region $R := \{(c,h)|0 < c < 1, h > 0\}$ leaving only sets of points given by eq. (115), i.e:

$$c = 1 - \frac{6}{m(m+1)} \quad \text{and} \quad h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \tag{116}$$

for $1 \leq r < m$ and $1 \leq s < r$. Hence the points in (116) that obey this condition are unitary in the forbidden region. These points are also referred to as first intersections and essentially all the points of the intersecting vanishing curves at the same level would be defined by (116). This completes the so-called nonunitarity proof.

4.5 Minimal models

The major success of two dimensional CFT at criticality is mostly due to the Kac determinant leading to conditions in eq.(116) which represents the set of minimal models (minimal because we have employed the algebra of local fields in deriving the former). In general minimal models reproduce exactly the scaling limit at criticality of lattice models in statistical physics systems. From [2] and [8], the critical exponent of the Ising model and the tricritical Ising model are described by the first series of minimal models, $\mathcal{M}(4,3)$ and $\mathcal{M}(5,4)$ along with the other models [1, 8]. We shall however concentrate on the former as they are central for the following chapter.

4.5.1 The Ising model and $\mathcal{M}(4,3)$

The Ising model is characterized by its spin observable σ_i and its energy density ε_i which are two local scaling operators at lattice sites i. The Ising model on a 2 dimensional lattice is therefore defined by the Hamiltonian

$$H = J \sum_{\langle i,j \rangle} \sigma_i \sigma_j + B \sum_i \sigma_i, \tag{117}$$

where J is a coupling constant, $\langle i,j \rangle$ represents sum over nearest neighboring sites and B is the external magnetic field which is set to zero for in our case. The shift from the ordered phase $\sum_i \sigma_i \neq 0$ to a disordered phase $\sum_i \sigma_i \neq 0$ characterizes a phase transition of second order described by the scaling. This phenomenon was studied using quantum theoretical methods in [31]. The critical exponents, η and ν are defined from the behavior of the 2-point correlators,

$$\langle \sigma_i \sigma_j \rangle \sim \frac{1}{(x_{ij})^{d-2+\eta}}, \qquad \langle \varepsilon_i \varepsilon_j \rangle \sim \frac{1}{(x_{ij})^{4-2/\nu}}, \tag{118}$$

where $d = 2$ is the number of dimensions of lattice and x_{ij} is the separation between two lattice sites i and j. Exact result initially derived by Onsager and Kauffmann sets $\eta = 1/4$ and $\nu = 1$.

We now make use of eq. (116) for $m = 3$ to evaluate the conformal dimensions and assume that both σ_i and ε_j are spinless. This leads $c = 1/2$ and $h_{1,2} = \bar{h}_{1,2} = 1/16$ and $h_{2,1} = \bar{h}_{2,1} = 1/4$. Under the operator-field correspondence [25] we relate the operators in (118) to their respective fields as

$$\sigma \Longleftrightarrow \phi_{1,2} \quad \text{and} \quad \varepsilon \Longleftrightarrow \phi_{2,1}, \tag{119}$$

such that the two-point correlators corresponding to $\mathcal{M}(4,3)$ theory read[29]

$$\langle \phi_{1,2}(z_i, \bar{z}_i) \phi_{1,2}(z_j, \bar{z}_j) \rangle = \frac{1}{|z_{ij}|^{2(h_{1,2}+\bar{h}_{1,2})}} = \frac{1}{|z_{ij}|^{1/4}}, \tag{120}$$

$$\langle \phi_{2,1}(z_i, \bar{z}_i) \phi_{2,1}(z_j, \bar{z}_j) \rangle = \frac{1}{|z_{ij}|^{2(h_{2,1}+\bar{h}_{2,1})}} = \frac{1}{|z_{ij}|^{1}}. \tag{121}$$

Hence the same information as for the Ising model is obtained from calculations involving algebra of local field. It can be noted that the data above also matches with what was initially derived as an exercise in eq. (107) in the operator sector for the 4-point function. The 2-point function is actually a limit of the 4-point function. We also have $\phi_{1,1}$ leading to $h_{1,1} = \bar{h}_{1,1} = 0$, This is associated to the identity field $\mathbb{I}$ which is related to the stress-energy tensor as $T(z) = \mathbb{I}^{-2}$ and its 2-point is known and is directly proportional to the central charge. This is further discussed in Ref. [1] (Ch. 18).

4.5.2 The tricritical Ising model and $\mathcal{M}(5,4)$

The next in the minimal model series is the tricritical Ising model occurring at $m = 4$ and has $c = 7/10$ and is described by the following Hamiltonian[30]

$$H = -J \sum_{\langle i,j \rangle} t_i t_j (K + \delta_{\sigma_i, \sigma_j}) + \mu \sum_i t_i - B \sum_i \sigma_i, \tag{122}$$

where t_i represents the possible vacancies in the lattice sites, K now is the energy of a pair of opposite spins while $K + 1$ represents the pair of like spins, μ is the chemical potential which measures the average number of occupied sites on the lattice and finally the last term has the exact same meaning as for eq. (117) [1]. The difference with the Ising model is that unoccupied sites are allowed number of spins on the lattice are allowed to fluctuate. The tricritical Ising has five scaling operators (three energy-like and 2 spin-like). At a given value of (β, K, μ) where $\beta = 1/T$ and T being the temperature, a critical point prevails where three phases coexists (hence the terms tricritical). From an examination of the Hamiltonian, one can infer that t_i as bosonic contributions and that

[29] We have combined the holomorphic and antiholomorphic sectors using the fact that the distances must be same between z_i and z_j and between $\bar{z}_i$ and $\bar{z}_j$ such that $|z_{ij}| = |\bar{z}_{ij}|$.

[30] Also known as the Blume-Emery-Griffiths model [32].

σ_i as fermionic contributions. At tricritical point, these two contributions are brought into equilibrium which is seen as spacetime supersymmetry present a minimal theory. This was probed in Ref. [28] in the context of topological superconductors[31].

This suggest that the tricritical Ising model require a more general action than just $S = \int d^2z\, \psi\partial_{\bar{z}}\psi + c.c$; a model that includes supersymmetry is required. The tricritical Ising model is best described in the framework of supersymmetric conformal field theories which can be consulted in Ref. [6, 15, 22]. The operators in the minimal model $\mathcal{M}(5,4)$ which describes the tricritical Ising model will eventually transform under a supersymmetric generalization of the conformal transformations and the algebra of fields and superfields close under the super Virasoro algebra (also known as the Neveu-Scharz and Ramond algebra [15]).

5 The Landau-Ginzburg theory

The Landau-Ginzburg theory is a different way of studying minimal models. A major section of this work concentrated on understanding the duality between statistical model at criticality and minimal theories with physical relevance identified as representations of the Virasoro algebra. However as noted in the various (introductory) prescriptions of the representation theory of $\mathfrak{V}$, one can easily loose track of the underlying physics because of non-Lagrangian description of the models. Fortunately a class of minimal theories (which resemble the set we derived in eq. (116) except that m is now just a bit more constraint as shown below) known as the diagonal unitary minimal models, $\mathcal{M}(m+1,m)$ does exist and are parametrized by

$$c_m = 1 - \frac{6}{m(m+1)} \qquad m = 2,3,4,... \tag{123}$$

$$h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \qquad 1 \leq r \leq m-1, 1 \leq s \leq m, \tag{124}$$

where r and s are defined similarly as for eq. (116). The primary fields of the theory are constructed from the left and right Virasoro representations (i. e: representations of $\mathfrak{V} \oplus \bar{\mathfrak{V}}$). The product of fields is then $\Phi_{r,s}(z,\bar{z}) = \phi_{r,s}(z) \otimes \phi_{r,s}(\bar{z})$ quoted from [1]. The simple effective Lagrangian description of this set of minimal model is

$$\mathcal{L} = \int d^2z \left[\frac{1}{2}(\partial\Phi)^2 + V(\Phi)\right], \tag{125}$$

where Φ is a self-interacting field and corresponds to the order parameter of the statistical system. The potential $V(\Phi)$ is usually a power-like potential invariant under the symmetry $\Phi \to -\Phi$ whose maximum values relate to several critical points of the system. This was initially discussed by A. B. Zamolodchikov [36] and the Lagrangian (125) is known as the effective Landau-Ginzburg Lagrangian. The most critical point of the system under consideration therefore is represented by the monomial

$$V(\Phi) = \Phi^{2(m-1)}. \tag{126}$$

Hence each $m-1$ minima is separated by $m-2$ maxima.

[31] A topological superconductor is characterized by gapped modes on the bulk and gapless modes on boundary protected by $\mathbb{Z}_2$ symmetry. This is explain further in section (6).

The important point that we understand from the Landau-Ginzburg theory is that the diagonal unitary models $\mathcal{M}(m+1, m)$ describing a multicritical point system can be represented in terms of a single scalar field. Hence only the information about a single scalar field is needed in the correlation functions which are then much easier to construct. This is just one of the few reasons why the latter is a slightly better method for describing minimal theories within the context of condensed matter systems, compared to the Coulomb gas method for example, where merits of the CFTs are more transparent. It can be shown that $\mathcal{M}(m+1, m)$ are indeed described by the effective multicritical Landau-Ginzburg theory using fusion rules. The details of the latter techniques including are discussed to greater lengths in Ref.[1] (Ch. 8). Figures (2) and (3) [37] illustrates the Landau-Ginzburg description of the universality classes at $m = 3$ and 4 respectively as shown below

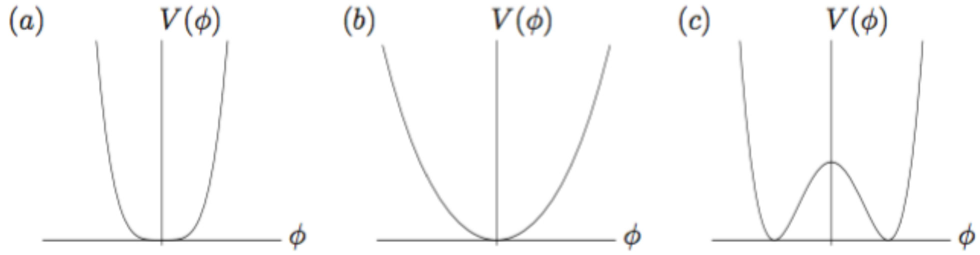

Figure 2: The Landau-Ginzburg potential $V(\phi)$ for the Ising model: (a) at criticality (b) when temperature T is large ($T >> T_c$) (c) at low tempatures $T \sim T_c$.

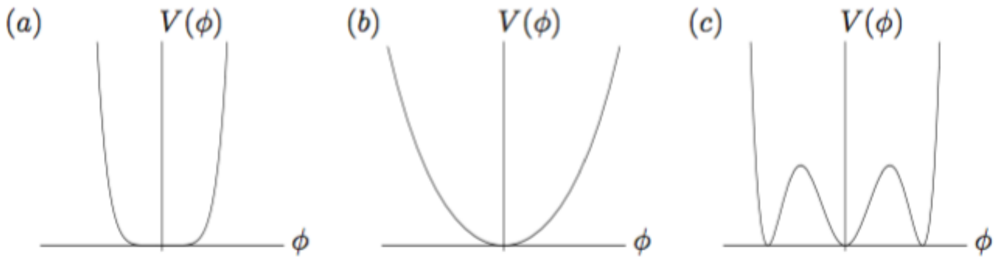

Figure 3: The Landau-Ginzburg potential $V(\phi)$ for the tricritical Ising model: (a) at tricritrical point (b) when T is large (c) when T is small such that essential subleading non-positive perturbations prevails. Images from Ref. [37].

These figures provides a more convenient description of the theory in (125) which can be easily bridged to concepts in statistical mechanics. Aside to providing such physical depiction of the minimal theories, the Landau-Ginzburg theory also allows one to perform perturbation of the CFTs away from critical points which can be associated to RG flows between different fixed point of different theories. Such flow is represented by substituting the potential V_m by $V_m + \alpha V_{m-1}$ where the regime when $\alpha \to 0$ corresponds to the $(m+1, m)$ fixed point and $\alpha \to \infty$ is associated with the fixed point $(m, m-1)$. However at criticality, as demonstrated in the earlier sections, behavior of the primary fields are best described using correlation functions and studying the divergence of the latter based on bootstrap methods provides greater insight from a mathematical point of view.

6 Emergence of supersymmetry at critical fixed point

In this chapter, we provide a brief account of the applications of conformal invariance to condensed matter systems motivated by the recent claims presented in Ref. [28, 35] that spacetime supersymmetry emerges in $d = 1 + 1$ in such systems. Supersymmetry is motivated by the numerous issues plaguing the standard model such as the hierarchy problem, the cosmological constant (Λ) problem and fine tuning of the standard model and some more issues. The systems under considerations are topological superconductors (TSCs) where fermions (a Fermi sea of electrons or helium-3) pair up in an unusual way. This results in states where fermions in the bulk possess an energy gap[32] however modes on the boundary (or surface)[33] are gapless. This suggests that a spontaneous quantum breaking of symmetry at (topological) phase transition will lead to gap modes. In doing so, spacetime supersymmetry at the critical point is found to emerge up to numerical accuracy in Ref. [28]. The gapless modes in the surface are protected by time reversal symmetry as indicated in Ref. [34, 35]. This result provide much insight between band structure and spacetime supersymmetry at criticality.

6.1 $d = 1 + 1, \mathcal{N} = 1$ emergent supersymmetry at boundary of TSC

In general terms, the system in consideration is the (1+1)-dimensional boundary of a time-reversal invariant (2+1)-dimensional topological superconductor [28, 34]. Before going into the details, we provide some explanation concerning the terminology. For example; A topological superconductor is a two or three dimensional entity with gapless modes on its boundary and gapped modes in the bulk. These kind of structures gained more attention after the discovery of the discrete $\mathbb{Z}_2$ symmetry associated with topology of superconductors and insulators, where $\mathbb{Z}_2 := \{1, -1\}$. One can refer to [34] for an in-depth review of the subject. This $\mathbb{Z}_2$ symmetry is what is termed at time reversal symmetry and transitions between one state (ordered)to the other (disordered) is referred to as a phase transition. This phase transition can occur in two and three dimensions and supports gapless Majorana modes which can be made to acquire a finite amount of energy (gapped mode) via spontaneous symmetry breaking of $\mathbb{Z}_2$ symmetry at the surface due to magnetization. In trying to understand the behavior and evolution of boundary modes under magnetization, one realizes the central charge of the tricritical Ising model emerges [28] and the latter is the only minimal theory endowed with supersymmetry [1, 2, 8, 15].

The model Hamiltonian which fits the above observation is [28],

$$H = -i \sum_i [1 - g\mu^z_{i+1/2}]\chi_i \chi_{i+1} + H_b, \tag{127}$$

where $\mu^z_{i+1/2}$ are the Ising spins sitting at the lattice sites i and H_b represents the interaction between the Ising spins on i and the magnetization h. χ_i represents a single

[32] A gapped mode is one where a finite amount of energy is required for excitation while a gapless mode is one where its energy tends to zero as momentum tends to zero such only an infinitesimal amount of energy is required for excitation. The latter is associated to criticality at phase transition. An example is the Goldstone mode which is a gapless excitation arising as a result of spontaneous symmetry breaking [24]. Hence for this reason the modes present in our case will be analogous to the Goldstino modes at the supersymmetric critical point.

[33] When $d = 2$ the bulk is surrounded by a one-dimensional boundary. When $d = 3$ the bulk is surrounded by a two-dimensional surface. Such distinction is crucial in studying this topic.

Majorana fermion at lattice site i with spin 1/2. Majorana modes on boundary are gapless due translation invariance. At low energies the system describe by eq. (123) bears the following general action formulation

$$S^{(d+1)} = \int d\tau d^d x \left[\frac{1}{2}\bar{\chi}\partial\!\!\!/\chi + \frac{1}{2}(\partial_\tau \phi)^2 + \frac{\nu_\phi^2}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 + g\phi\chi\bar{\chi} + u\phi^4\right], \tag{128}$$

where $d = 1$ in the present case, χ represents the Majorana field (i.e: the Ising fermionic degree of freedom), ϕ represents the Ising field (i.e: the Ising bosonic degree of freedom) and the usual Dirac and Pauli matrices were used similarly to the ones in section (2.3). When h is large, the Ising spins are disordered and when h is small, the Ising spins are ordered leading to gapped Majorana modes[34]. As described in Ref. [28] for a particular value of h, criticality is reached which is denoted by h_c separating the gapless ($h > h_c$) and the gapped mode ($h < h_c$) of the Majorana fermions, the value of the central charge obtained is $c = 7/10$ as theoretically indicated in Ref. [8, 15]. This is the central charge associated to the $\mathcal{M}(5,4)$ minimal theory which describes the tricritical Ising model.

However an interesting issue that can be immediately identified in the above model is the fact that the mass term $r^2\phi^2/2$ of the model is not zero at criticality and still the minimal theory $\mathcal{M}(5,4)$ prevails when $h = h_c$. This is because we know from earlier discussions that mass terms break conformal invariance and hence the former have to be absent at criticality if an accurate description of the statistical system is to be achieved. This is a problem which can be further studied through more formal treatments. Extending the above model to $d = 2+1$ presents some further issues which is the fact that the theory is a low energy theory and it is not trivial to understand how spacetime supersymmetry is realized in such framework. It can be noted that if $\nu_\phi^2 = 1$, $m^2 = r/2$ and $u = \lambda$, one ends up with the Gross-Neveu (-Yukawa) model as described in Ref. [27] (Pg. 16, Eq. 4.4 and Eq. 4.5). One can then ask what would then be the fixed points of the Gross-Neveu (-Yukawa) model but the latter is describing a 3D CFT - which makes analytical evaluation of such data with precision a highly non-trivial task. (Some results were derived computationally in Ref. [28] (App. B and App. C) using Wilsonian RG techniques).

[34]The similarity with statistical mechanical critical phenomena in ferromagnetism can be noted [1].

7 Conclusions and outlook

In this thesis, we were primarily interested in understanding the general structure of conformal theories and representation theory of the infinite dimensional Virasoro algebra in two dimensions and how statistical systems are reproduced from such representations. A concise study of these concepts has allowed us to construct the minimal models and calculate the critical exponents for the universality class of the two-dimensional Ising model and discuss that of the tricritical Ising model in chapter 4. The techniques discussed are all very well known and well established in the vast array of literature on conformal field theory and related topics. From chapter 2 to 5 an effort was made to be as original as possible in the discussions without straying to far within the rich mathematical details developed other the years especially in the area of representation theory. In the final section of this work, we very briefly consider the claims presented in the paper by Grover et al. in Ref. [28]. The latter as it can be noted is not as intellectually gratifying as one would want it to since the topic of emergent supersymmetry is a relatively new one. However some recent frameworks was proposed by L. Iliesiu et al. in Ref. [27] which involve the bootstrap method for fermions in $d = 3$ which seems to be a rather promising approach based on the stronger bounds derived from numerical results presented in [27](Pg. 20).

A detailed discussion of this work goes beyond the scope of this thesis subject and is not possible due to time constraints. It seems to be worth to further analyzed the model in eq. (128) [28] using the three dimensional fermion bootstrap and deriving the critical exponents with much greater exactitude. This shall be the subject of further research; to try an understand the conjecture of spacetime supersymmetry emerging in $d = 1 + 1$ at the boundary of a topological superconductor and also explore cases when $d > 2$. We hope to have provided just a concise discussion to the area of conformal field theory and its potential application to the study of emergent supersymmetry. The latter is a recent topic and a very interesting application of conformal symmetry at phase transition with signatures strong enough (at least in the two-dimensional case as discussed in [28, 32, 37]) that the subject is worth studying and theoretical frameworks proposed to study the subject in greater details are equally crucial.

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