Dynamics of an Inflationary Universe

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Abstract

The initial conditions needed by the standard cosmological model lead to precarious features. This motivates the search for a completely new picture of the early universe which is the inflationary scenario. Contained within this report is a study of the standard cosmological model followed by the development of the paradigm of inflation within the context of chaotic and standard model Higgs inflation. The phenomenon of reheating is examined as a two stage process; preheating and thermalisation. Normalization of the inflationary models according to data from the cosmic microwave background anisotropy is discussed. The theoretical predictions and experimental values of the cosmological parameters are then compared.

Contents

1 Introduction

Standard cosmology rest solely on the ideas developed by Friedman, Robertson and Walker [1 - 3, 5 - 9, 16, 19 - 21, 24] (FRW cosmology).This standard cosmological framework represents towering success providing information about the evolution of the universe with a great deal of precision and accuracy [1, 16, 19]. However this model is plagued by various issues, many of which related to the initial conditions [1, 5]. Such puzzles therefore demands a more complete description of the energy density sources in the universe leading to the construction of an entirely new picture of the early universe. This is the paradigm of inflation. Inflation can be referred to as an epoch of accelerated expansion of the early universe; a phase during which the scale factor grows as per an exponential law. This phase is also known as the de Sitter phase [4].

The idea of inflation originated initially from Guth (1981) [4] as a proposal to resolved the puzzles of standard cosmology. This model of old inflation was later improved upon by Linde' s [5, 9] chaotic inflation and Albrecht and Steinhdart [8] offering more accurate treatment of the first-order phase transitions within the framework of standard grand unified theories as discussed in Ref. [2]. These ideas led the way for new inflationary cosmologies with interesting predictions. The inflationary scenario by Guth and others not only provides solutions to the many fundamental problems of FRW cosmology which will be discussed in the following section but also addresses the question of density perturbations which display nearly scale invariant spectra [19, 24].

Fortunately high precision cosmology has rendered it possible to investigate the nearly scale invariant spectrum of density perturbations through measurements of the temperature anisotropies in Cosmic Microwave Background (CMB) radiation. These very first predictions of cosmic inflation was observed by the Cosmic Background Explorer in 1992 and is backed up by many other experiments [19, 24 - 26] such as the Wilkinson Microwave Anisotropy Probe (WMAP) with much higher accuracy.

The report is organised as follows; in the second chapter 2, the shortcomings of the hot big bang model is introduced through the flatness problem,the horizon problem and the relic density problem are introduced and explained. This shall allow one to foresee the weaknesses of FRW cosmology.The third chapter concentrates on the paradigm of inflation through the theory of a single scalar field, the inflaton [2, 19, 23] and the parameters which dictates the slow rolling of the field down the potential, $V(\phi)$. Chapter 4 introduces the inflationary models, namely chaotic inflation with ϕ^2 and ϕ^4 and the standard model Higgs inflation non-minimally coupled to gravity. Chapter 5 discusses the theory of reheating, the stages after inflation; that is preheating and thermalisation. The anisotropies of the cosmic microwave background are studied in chapter 6 by considering the spectral indices and the tensor-to-scalar ratios derived theoretically from the inflationary models discussed in this report and compares them with actual experimental data.

2 Shortcomings of the Standard Big Bang Model

2.1 Standard Cosmology

Cosmology as the utilization of the framework of the general theory of relativity would seem an insurmountable enterprise were it not for an extraordinary fact - the universe is isotropic and spatially homogeneous on evening out over the largest scales [1, 2]. This last statement is referred to as the cosmological principle. The study of the standard cosmological model is based on the fact that the universe looks the same in all direction (isotropy) and at every point (homogeneity). From the argument of isotropy and purely geometrical facts, as described in Ref. [1], the three dimensional space can be characterized by the metric of the Robertson-Walker form:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2(\theta))d\phi^2\right], \tag{1}$$

where $a(t)$ represents the scale factor and t the cosmic time [31]. The constant, k, describes spatial curvature where positive, zero and negative values describe section of space which corresponds to respectively closed three dimensional space, local flatness and a locally hyperbolic space. It must be noted that the above equation is consistent with the description of spacetime as a smooth manifold as per the formalism of general relativity and the cosmological principle [1, 4].

The evolution of the universe, more precisely the evolution of the scale factor $a(t)$ is governed by the solutions of the Einstein fields equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2}$$

where $R_{\mu\nu}$, R, G, $g_{\mu\nu}$, $T_{\mu\nu}$ are the Ricci tensor, the Ricci scalar, the gravitational constant, the metric tensor and the energy-momentum tensor respectively. In order to proceed further in understanding the material present within the universe at given epochs, it is firstly useful to identify the energy-momentum tensor as a perfect fluid [1] and therefore reach a conservation equation from which the equation of states can be obtained. The energy-momentum tensor, $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)U_\mu U_\nu, \tag{3}$$

where U_μ represents the fluid four-velocity, ρ is the energy density and p is the pressure all defined within the inertial (rest) frame. With the above definition and equation (1), the Friedmann equation can be obtained [1, 16, 27] and reads,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}, \tag{4}$$

where H^2 is the Hubble parameter. The acceleration equation then follows immediately from above definition

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{5}$$

Equation (4) and (5) are then combine to yield the energy conservation equation. This procedure can also be reproduced using the vanishing covariant divergence of the energy momentum tensor as shown in Ref. [27]. The fluid equation therefore reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{6}$$

The fluid equation signifies that local changes in the energy density are induced because of the expansion of the universe (as understood by H). There is an interchange of energy between matter geometry of spacetime. This is an obvious statement since one starts from the Einstein fields' equation which clearly describes such relation between the evolution of matter and spacetime curvature. The equation of states for a radiation and a matter dominated are obtained respectively as

$$p = \frac{\rho}{3} \tag{7}$$

$$p = 0. \tag{8}$$

These equations plays key roles in understanding the evolution of the universe based on the material present within it and this can be seen after explicitly writing down solutions of equations (4) and (6).

For a radiation dominated universe, the solutions are

$$a(t) \propto t^{\frac{1}{2}}, \tag{9}$$

$$\rho \propto a^{-4}. \tag{10}$$

For a dust dominated universe, the solutions are

$$a(t) \propto t^{\frac{2}{3}}, \tag{11}$$

$$\rho \propto a^{-3}. \tag{12}$$

The framework of the standard cosmological model as described by the equations above appears to be rather consistent with the present day observations, however it should be noted that standard cosmology has not given any explanation for a homogeneous and isotropic universe. It cannot account for the issues related to the evolution of the early universe that severely contradict actual cosmological data. These issues are examined in the following sections.

2.2 The Flatness Problem

The large scale observable universe as seen today appears very flat and very homogeneous. This can be explicitly worked out by writing the Friedmann equation in terms of the density parameter, Ω. From equation (4), one finds

$$\Omega = 1 + \frac{k}{a^2 H^2}, \tag{13}$$

where Ω is expressed as the ratio of the energy density, ρ to the critical density, ρ_c where the critical density is defined as

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{14}$$

Therefore from the understanding of standard cosmology, $\dot{a}$ is decreasing with time meaning that Ω is deviating from one according to equation (13). Hence the question that this raises is that how is it possible that Ω can be so close to one today if theory indicates that it has been divergence from unity for about ten billion years?

The solution to this question would that Ω was incredibly close to unity suggesting some extreme fine tuning. Ω would be accurate up to an order of 10^{-16} at the epoch of nucleosynthesis as shown in Ref. [24]. This would make the universe to either collapse or expand very rapidly such that in both cases conditions are far from optimum for formation of large scale structures. This is the flatness problem. However if the inflationary scenario is applied to the problem, one finds that Ω tends toward one during the inflationary phase of the very early universe and its value does not diverge even within this present epoch.

2.3 The Horizon Problem

The temperature anisotropy of the Cosmic Microwave Background radiation suggests clear causal disconnection between distant patches of the dark sky. Hence how is it possible for two regions of the dark sky that have never been in causal contact yet have the same temperature up to actual experimental accuracy? To answer this question the difference between the scale of the particle horizon and the comoving particle horizon is pointed out in order to find causally connected regions in the early universe.

The particle horizon (physical) distance is given as

$$D_H(t) = a(t) d_H(t). \tag{15}$$

The comoving particle horizon is

$$d_H(t) = \int_{t_*}^{t} \frac{dt'}{a(t')}, \tag{16}$$

where t_* is set to 0. The physical horizon, D_H is calculated to be $2H_*^{-1}$, therefore the physical horizon today is given as $2H_0^{-1}$. The comoving horizon distance since time of decoupling is $\backsim 10^{-2}$ giving a ratio [1] of 10^{-2}. This indicates that regions that are at the present time in complete causal disconnection were in the past a much smaller fraction of the Hubble' s radius. As a result only a very small small section of the CMB should show this causal connection. However as indicated by modern day cosmological data, the universe has the same temperature everywhere to a very high precision. This suggests a modification of the causal configuration of the conventional standard cosmological model based on FRW.

In an inflationary model, the physical horizon would be "pushed" out the Hubble radius during the rapid inflationary phase. The Hubble radius would then grow faster than the physical horizon. These statements based hence motivates the following condition

$$\int_{t_{dec}}^{t_*} \frac{dt}{a(t)} \gg \int_{t_0}^{t_{dec}} \frac{dt}{a(t)}, \tag{17}$$

where t_{dec} is the time of decoupling which corresponds to the surface of last scattering, the earliest measurements of photon scattering that present day experiments are able to measure.

2.4 Relic Density Problem

Within grand unified theories the local symmetry is spontaneously broken at an energy $M \approx 10^{16}$ GeV to the gauge symmetry of the standard model of particle physics with gauge group

$SU(3) \times SU(2) \times U(1)$ [2]. Such cases of broken symmetries occurs successively during the phase transitions leading to formation of topological defects. The latter are classified according to the structure of the symmetry breaking pattern. Hence one configuration of this symmetry breaking pattern lead to formation of monopoles via the Kibble mechanism [4]. If these massive particles prevailed during the early moments of the universe then their energy densities would vary as ρ^{-3} and if stable would come to dominate the radiation dominated era (which goes as ρ^{-4} leading to very different conditions. An inflationary period, with $\ddot{a} > 0$ within the early universe, if lasts long enough should "pushed" the scalar field outside the Hubble radius such that phase transitions occurs outside the Hubble radius and therefore unwanted relics such as primordial monopoles are diluted away.

3 Scalar Field Inflation

Within this framework of new cosmology the requirement is to have the scale factor to expand exponentially. Using the Friedmann and acceleration equation, the following relation can be obtained

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = 8\pi G p. \tag{18}$$

The scale factor grows as $a(t) \backsim e^{tH}$ during the inflationary phase (or de Sitter phase). The Friedmann equation followed by the expression for $a(t)$ are substituted in the above equation in order to yield the equation of state of matter [4]

$$p = -\rho. \tag{19}$$

Hence in order to generate inflation one requires a field theoretic description of matter. The field that fits such description is simply a single scalar field, $\phi(x_i, t)$, where ϕ is referred to as the inflaton which couples minimally to gravity

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\right], \tag{20}$$

such that the Lagrangian density of the scalar field (not considering the gravity part of the action), ϕ reads

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi - V(\phi), \tag{21}$$

where $\mathcal{R}$ is the four dimensional Ricci scalar, $g = \det(g_{\mu\nu})$, ϕ is a real scalar field and $V(\phi)$ is the potential.

The energy-momentum tensor [1 - 3], $T_{\mu\nu}$ is obtained by varying the action with respect to $g_{\mu\nu}$

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) + V(\phi)\right], \tag{22}$$

where $g_{\mu\nu}$ is the metric tensor with signature (+, -, -, -). Since the energy-momentum tensor was previously assumed to behave a a perfect fluid so will the homogeneous real scalar field. Hence the components of energy density and pressure of $T_{\mu\nu}$ are substituted within the fluid equation to yield

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{23}$$

which is the equation of motion of the scalar field.

One now consider $\dot{\phi} \ll V(\phi)$, this implies that the potential energy of the scalar field is the principal contribution to both ρ and p. This results in equation (19). Therefore within this approximation $p \simeq -\rho$. This is quite similar to when the universe is dominated by an energy component leading to an exponential expansion where this energy component plays almost the role of a cosmological constant. Hence neglecting the kinetic energy allows the field to slowly roll down its potential.

3.1 Slow Roll Phase

The slow rolling of the single scalar field down the potential is governed by the slow roll conditions [1 - 3]. These are obtained by assuming that for inflation to occur, the potential, $V(\phi)$ must dominate the kinetic energy. This also further means an adequately flat potential for inflation.

$$3H\dot{\phi} \simeq -\frac{dV}{d\phi}, \tag{24}$$

$$3H^2 \simeq 8\pi G V(\phi). \tag{25}$$

The above relations are satisfied by the following which within the FRW framework ensure cosmic acceleration and a sufficiently long period of time for inflation to persist.

$$|\epsilon| \ll 1, \tag{26}$$

$$|\eta| \equiv \frac{|\dot{\epsilon}|}{H\epsilon} \ll 1, \tag{27}$$

where $\epsilon = -\frac{\dot{H}}{H^2}$. A given potential, $V(\phi)$, is therefore suitably assessed through the evaluation of the slow roll parameters whose smallness guaranty slow roll inflation. These parameters are defined as

$$\epsilon = \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 \tag{28}$$

$$\eta = M_p^2 \frac{V''}{V}. \tag{29}$$

The above equations can be substituted back in equation (23) and (24) to give inflation, the condition $\frac{\ddot{a}}{a} > 0$. This can be therefore be used to evaluate how much inflation occurs as shown. The end of inflation occurs when equations (25) and (26) are of order of unity. This is does not however implies reheating of the universe.

Using the slow roll conditions, the amount of inflation or number of e-foldings can be evaluated by simply diving (24) by (23) and integrating both sides.

$$\mathcal{N} = ln\left(\frac{a(t_{end})}{a(t_i)}\right) = \int_{t_i}^{t_{end}} H dt = -\frac{8\pi}{M_{pl}^2}\int_{\phi_i}^{\phi_{end}} \frac{V(\phi)}{V'(\phi)} d\phi, \tag{30}$$

where the subscript i and end represents the beginning and end of inflation respectively. The above argument is applied to the flatness problem. Ω is required to be $|\Omega - 1| \leqq 10^{-60}$ just after inflation. Using the expression for Ω from equation (13), one obtains

$$\left(\frac{a_i}{a_{end}}\right)^2 = e^{-2\mathcal{N}}, \tag{31}$$

where $\mathcal{N}$ must appropriately be $\gtrsim 60$.The number of e-folds which solves the horizon problem is also found to be above 60. Hence this establishes a standard lower bound for the number of e-folds, where $\mathcal{N} \gtrsim 60$, which must satisfy an inflationary scenario.

4 Inflationary Potentials

The publication of Guth' s paper (1981) [5], as a proposal to answer the question of how the total energy density of the universe was dominated by the potential energy of a scalar field, sparked a growing interest among cosmologists in understanding the inflationary scenario. This has led to the elaboration of numerous models [25] coming from different areas of theoretical particle physics and cosmology. The lack of means to perform astronomical observation beyond the observable universe in order to verify the predictions of many models of inflation has motivated the search for models of cosmological inflation that arise from particle physics framework. One of the first such model was the Coleman - Weinberg Inflation [2, 25] which was related to the breaking $SU(5)$ symmetry group. A theory of inflation derived from particle physics would be a very complete one, unfortunately the Coleman - Weinberg model had some issues and was supplanted by other models.

The models of inflation are classified according the number of constraints (order of parameter) imposed on the scalar fields [25]. These are parameters can be verified from cosmological observations [25]. These parameters, which constraints the data from measurements of the cosmic microwave background anisotropy, are powerful tools in understanding whether a model is on the right track or not. One can ask why are single scalar field studied, the simplest answer is that they are not yet ruled out! Planck 2013 [19, 21, 22] showed that perturbation of non - Gaussian nature were non existent. This is because such perturbation would be the result of multi-field inflation. The models which will be investigated in this section is this section are; Linde' s Chaotic inflation [9] for a ϕ^2 and ϕ^4 scalar field and the standard model Higgs Inflation [25].

4.1 Chaotic Inflation with ϕ^2 and ϕ^4

Choatic inflation, also referred to as Large Field Inflation (or Mixed Large Field Inflation which is a generalisation [25]) implies that the field value before inflation can have any value, it is random (chaotic). The field is considered to be large at the start of inflation and rolls down the potential to smaller field value hence the name Large Field Inflation. The potential that is considered is the following

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4, \tag{32}$$

where λ is constraint imposed on the field. The potential above is redefined such that it is expressed in terms of M^4. This is easier to obtain the parameters present in the model in the amplitude for direct evaluation from CMB data. This also makes it easier to write down the mass of the inflaton [25].

$$V(\phi) = \frac{M^4}{M_{Pl}^2}\phi^2 + \alpha\frac{M^4}{M_{Pl}^4}\phi^4, \tag{33}$$

where M is the mass of the inflaton and $\alpha > 0$ is a dimensionless parameter.

The regimes that are considered are $\frac{\phi}{M_{Pl}} \ll \frac{1}{\sqrt{\alpha}}$ and $\frac{\phi}{M_{Pl}} \gg \frac{1}{\sqrt{\alpha}}$. However one can start by considering an interesting case where $\frac{\phi}{M_{Pl}} \simeq \frac{1}{\sqrt{\alpha}}$ and then work through the different regimes formerly stated. For easier manipulation of the arithmetic, the substitution $y = \frac{\phi}{M_{Pl}}$ is made, where $M_{Pl} = 2 \times 10^{19}$ GeV is the Plank mass [25]. Using equation (28) and (29) the calculation of the slow roll parameters leads to

$$\epsilon = \frac{2}{y^2}\left(\frac{2\alpha y^2+1}{\alpha y^2+1}\right)^2, \tag{34}$$

$$\eta = \frac{4}{y^2}\left(\frac{2\alpha y^4+\alpha y^2+1}{\alpha y^2+1}\right). \tag{35}$$

The integration of equation (30) with respect to the potential, $V(\phi)$ of chaotic inflation is performed to give an estimation of the number of e-folds.

$$\mathcal{N}_{end} - \mathcal{N} = -\frac{1}{8}\left[y_{end}^2 + \frac{1}{2\alpha}\ln(2\alpha y_{end}^2+1) - y^2 - \frac{1}{2\alpha}\ln(2\alpha y^2+1)\right]. \tag{36}$$

The above expression is then studied in both regimes and it can be seen that in the case of where $\alpha \ll 1$ we have

$$\mathcal{N}_{end} - \mathcal{N} = \frac{y^2 - y_{end}}{4}, \tag{37}$$

when $\alpha \gg 1$ it therefore be inferred that $\lambda \ll 1$ such that

$$\mathcal{N}_{end} - \mathcal{N} = \frac{y^2 - y_{end}}{8}. \tag{38}$$

so second inflation is half as less as first one and it can also be seen that the field value at end of inflation is smaller than field value at beggining of inflation.

The value $\phi_{CMB} \sim 15M_{Pl}$ [16] is inserted in equations (36) and (37), the number of e-folds obtained are; $\mathcal{N} = 56$ and $\mathcal{N} = 28$ for ϕ^2 and ϕ^4 respectively. The total number of e-foldings is $\mathcal{N} = 84$ which is large enough to solve horizon, flatness problem and the monopole problem. Using information from Ref. [25] the mass of inflaton is calculated to be $\frac{M}{M_{Pl}} \simeq 10^{-3}$ which lies perfectly within the GUT scale where $M_{GUT} = 2 \times 10^{16}$ GeV.

4.2 Standard Model Higgs Inflation

The actual action is written in the Jordan frame [25] where such a frame is defined as one where a scalar field is coupled to the Ricci scalar (a curvature scalar). In this case the Higgs field, H, is non-minimally coupled to gravity. In order to render the calculations easier it is a good thing to decouple the field term from the Ricci scalar. This is done by transforming to the Einstein frame using conformal transformations (Weyl rescalings [27]) as discussed in Ref. [25]. After transformations the action now reads

$$\mathcal{S} = 2M_{Pl}^2\int d^4x\sqrt{-g}\left[\frac{\mathcal{R}}{4} - \frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \mathcal{W}(\chi)\right], \tag{39}$$

where $\mathcal{W}(\chi)$ is the potential defined in the Einstein frame. At this point one can also defined ϕ to equal to $\sqrt{2}M_{Pl}\chi$ and $V = \mathcal{W}/2M_{Pl}^2$ The field χ is evaluated and approximated as

$$\chi \simeq \frac{\sqrt{3}}{2}\ln(1+\xi h^2), \tag{40}$$

where ξ is a dimensionless parameter employed in the transformations from Jordan to Einstein frame.

The potential in the Einstein frame can be written as (following the definitions of the transformations employed)

$$V(h) = \frac{\lambda M_P^4 l}{4}\left[\frac{h^2 - \nu^2}{1 + \xi h^2}\right]^2, \tag{41}$$

where ν is the Higgs vacuum expectation value (vev) which is equal to 175 GeV [25]. The substitution for h in terms χ using equation (40) and the relation between χ and ϕ are used to obtain the final form of the potential which reads

$$V(\phi) = \frac{\lambda M_P^4 l}{4\xi^2}\left[1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}}\right]^2 \tag{42}$$

as seen mass scale is entirely determined by the amplitude where ξ is the same dimensionless parameter previously mentioned and λ is a self-interacting term. Hence since all the parameters are contained within the amplitude, the standard model Higgs inflation is therefore a zero-parameter model. The construction of the shape the potential, $V(\phi, \chi)$ in the Einstein frame is shown below [31]

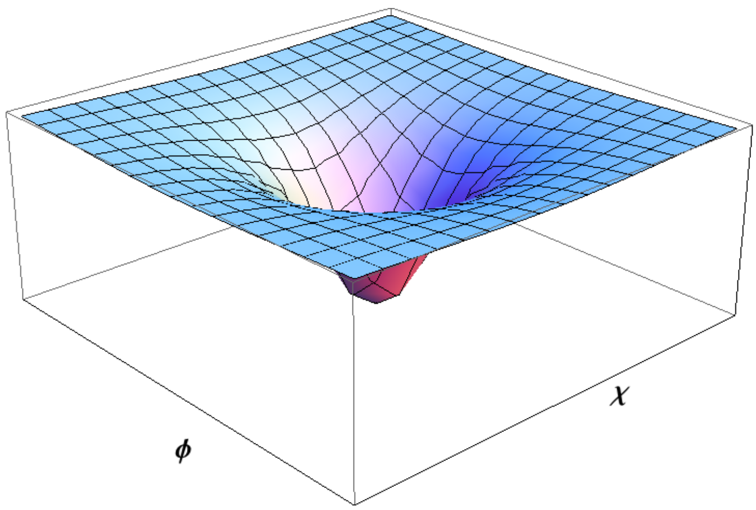

Figure 1: The standard model Higgs potential, $V(\phi, \chi)$, in the Einstein frame. This representation of equation (42) clearly shows how the potential flattens at large field values.(Credit: Ref. [31]).

This flattening of the potential at large field values is an important characteristic in the standard model Higgs inflation. This implies that the contribution from the Higgs self-interaction can be neglected since it is exponentially suppressed as understood from equation (42). Furthermore considering the amplitude of the potential in equation (42), one has λ/ξ^2 that helps fixing the normalization of the cosmic microwave background. Since it is not simply the scalar self-coupling λ, this makes the standard model Higgs to behave properly as the inflaton.

The slow parameters are

$$\epsilon = \frac{4}{3}\left[1 - e^{\sqrt{\frac{2}{3}}y}\right]^{-2} \tag{43}$$

$$\eta = \frac{2}{3}\left[\sinh\left(\frac{y}{\sqrt{6}}\right)\right]^{-2}, \tag{44}$$

Using integral from equation (30) but this time with respect to the Higgs potential (in the Einstein frame (Eq. (42)), the number of e-folds are calculated as

$$\mathcal{N}_{end} - \mathcal{N} = \frac{1}{2}\sqrt{\frac{3}{2}}(y_{end} - y) - \frac{3}{4}\left(e^{\sqrt{\frac{2}{3}}y_{end}} - e^{\sqrt{\frac{2}{3}}y}\right), \tag{45}$$

we evaluate the above in the regime where $\frac{\phi}{M_{Pl}} \gg 1$ such that last term is the dominant one (since an expoenential function grows or decays faster than any other functions). Therefore $\mathcal{N}$ for the Higgs inflation is between 55 and 59 [25, 27]. From Ref. [25] the $\frac{M}{M_{Pl}} = 4 \times 10^{-5}$ which again falls right within the interval of the GUT scale. With this prescription the mass of the Higgs boson can be calculated to be 125 GeV using the fact that $m_H = \nu\lambda$.

5 Theory of Reheating Beyond Inflation

5.1 Reheating of Homogeneous Inflaton Field

Within the realm of the inflationary scenario , the process of reheating is fundamental to particle creation and hence to the evolution of the standard hot big bang model. Inflation terminates when the potential gets steeper such that the inflaton acquires kinetic energy. This energy is thus transferred to standard model particles through decay of the inflaton field, ϕ [28, 29]. This process is known as reheating. The latter is discussed for the large fields inflation potentials (ϕ^2 and ϕ^4) for relatively homogeneous transition dictated by the following Klein Gordon equation previously seen,

$$\ddot{\phi} + 3H\dot{\phi} + V^{'}(\phi) = 0, \tag{46}$$

where $V(\phi)$ will firstly be represented by the quadratic potential, $\frac{1}{2}m^2\phi^2$. With the Hubble parameter defined as

$$H^2 = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2\right), \tag{47}$$

where M_{Pl} is the Planck Mass.

For a quadratic potential the solution is calculated to be

$$\phi(t) \approx \Phi(t)\sin(mt), \tag{48}$$

where $\Phi(t) = \frac{M_{Pl}}{mt}$ and m is associated to the frequency of the oscillation of the homogeneous inflaton field. Hence this implies that the energy density of the inflaton oscillating within a quadratic potential corresponds to a pressureless (dust) Universe. This is the matter dominated era.

$$\rho_\phi \sim a^{-3}. \tag{49}$$

The quartic potential is subjected to a similar treatment as above where this time $V(\phi) = \frac{1}{4}\lambda\phi^4$. The final result for the energy density, ρ_ϕ is

$$\rho_\phi \sim a^{-4}. \tag{50}$$

These are results for a homogeneous inflation of two distinct periods of a matter dominated era and a radiation dominated era.

In the former calculations, the coupling of the inflaton to some standard model particle have been ignored such that the decay rate denoted by Γ is much smaller than the Hubble parameter [28]. However in order to prevent an empty Universe [16] it is important that the energy from the inflaton goes in the decay process of the latter into a standard model boson, χ.

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V^{'}(\phi) = 0, \tag{51}$$

where Γ represents the decay rate of the inflaton into the field χ which is analogous a friction term added to the homogeneous inflation field. Therefore during this period one assumes that the expansion of space does not occur [29]. This is an important assumption that will help elaborate the following subsection by noting that reheating is not a single stage process because of how Γ parametrizes the decay rate of the inflaton.

5.2 Preheating and Thermalisation

Preheating can be defined as a rapid decay of the inflaton into bosons through a mechanism known as parametric resonance as discussed in Ref. [16]. This stage is called preheating because the bosons produced are not in thermal equilibrium. The inflaton is now considered to be coupled with a standard model boson, χ through a dimensionless coupling constant g. The presence of such couplings, especially via gravitational interaction as indicated in the section 4.2 where the standard model Higgs potential is discussed ,favor the study of a preheating stage through decay to standard model particles.

A simple interaction term Lagrangian [16] with ϕ coupled to χ can be written as

$$\mathcal{L} = -\frac{1}{2}g^2\phi^2\chi^2. \tag{52}$$

In order to understand briefly particle creation in classical inflation background the quantum theory of the χ particle production is subsequently analysed. The standard model field, χ is expanded into the creation and annihilation operator, $a_k^\dagger$ and a_k respectively via a three dimensional Fourier transform.

$$\chi(t, \mathbf{x}) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k(\chi_k^* a_k e^{i\mathbf{k}.\mathbf{x}} + \chi_k a_k^\dagger e^{-i\mathbf{k}.\mathbf{x}}), \tag{53}$$

where χ_k is the quantum field in the momentum space, $d\mathbf{k}$ and k is the momentum. From the above equation and the definition of the amplitude of the oscillation of the inflaton, Φ in (48), the following differential equation is obtained,

$$\ddot{\chi_k} + [k^2 + m_\chi^2 + g^2\Phi^2 \sin(mt)]\chi_k = 0. \tag{54}$$

Now from equation (54), the Matthieu equation [16, 28, 29] can be easily derived. This is done by firstly considering a substitution, $z = mt$ and rearranging the equation with respect to z. This implies, $\ddot{\chi_k} = m^2\chi''_k$ and $\dot{\chi_k} = m\chi'_k$. Using the final relation $\cos(2z) = 1 - 2\sin^2(z)$, one obtains the Matthieu equation which reads:

$$\chi''_k + [A_k - 2q\cos(2z)]\chi_k = 0, \tag{55}$$

where $A_k = \frac{k^2 - m_\chi^2}{m^2} - q$ and $q = \frac{g^2\Phi^2}{4m^2}$. The solutions of the Matthieu equations are very interesting by providing the appropriate tools for understanding the mechanism of parametric resonance which in turns explains how particle production occurs from decay of the inflaton. It also helps analysing the regions of stabilities and instabilities of parametric resonance.

After the preheating stage Thermalisation [16, 28, 29] takes places which is the period where particle can interact with each other creating other particles leading to decoupling (or Freezing) where the equilibrium between matter and radiation is lost. The photons which radiate through the universe form the background radiation field observed today.

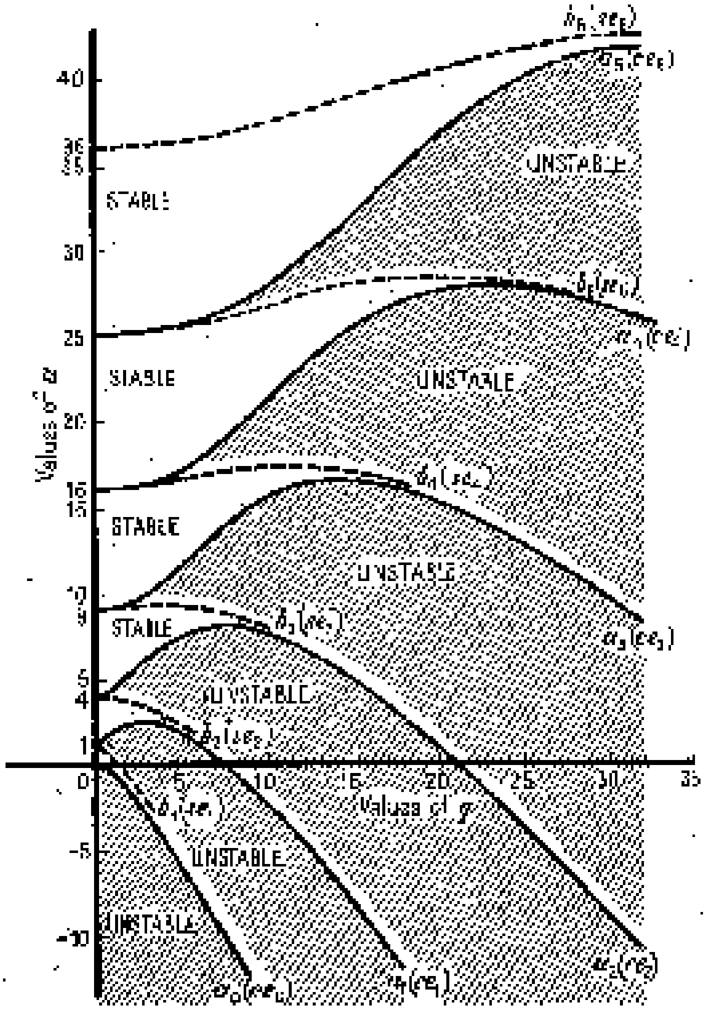

Figure 2: Band instability representations from solutions of Matthieu equations where the horizontal axis is the parameter q and the vertical axis represents the value of A_k. Efficient parametric resonance occurs when $q \gg 1$. (Credit: Ref [29]).

6 Study of the Cosmic Microwave Background

6.1 A (very) Brief History of the Primordial Background Radiation

The discovery of the primordial electromagnetic radiation in 1965 has shed much light on about the dynamics of the early universe and has hugely motivated research in high precision cosmology. The primordial background radiation, also referred to as the cosmic microwave background radiation (CMB) is a relatively isotropic black body such that its energy density as given as per Planck' s law

$$u(\lambda, T) = \frac{\lambda^{-5}}{\exp(-\frac{h\nu}{k_B T}) - 1}, \tag{56}$$

where h is the Planck' s constant, λ is the wavelength of the CMB, k_B is the Boltzmann constant, T is the temperature and ν is the frequency associated to the wavelength, λ. The total energy is therefore given as

$$U = \int_0^\infty d\lambda u(\lambda) \propto (k_B T)^4. \tag{57}$$

Such that the spectrum of the radiation field plotted; intensity against wavelength (obtained from statistical mechanics) is given as

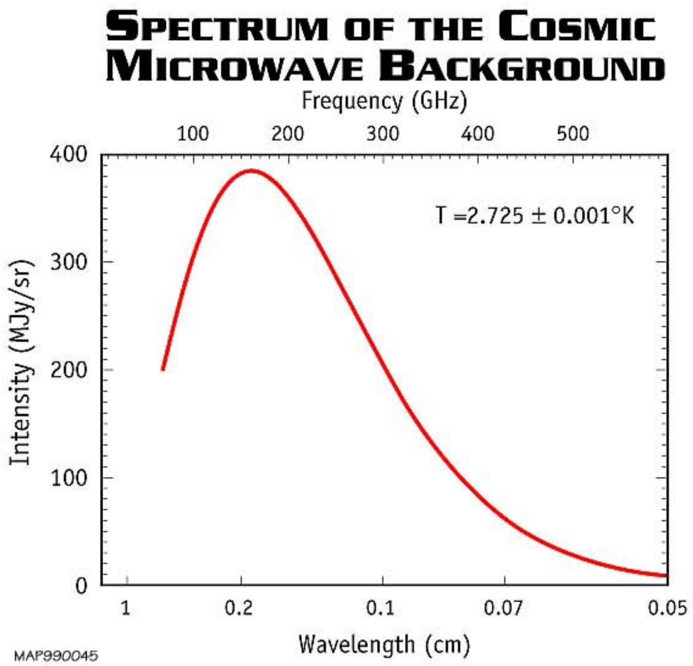

Figure 3: The spectrum follows the Planck' s distribution law with measured temperature, $T = 2.725 \pm 0.001$K. (Credit: NASA Science Team.)

It can be noted that the reason the CMB behaves as a blackbody even if the universe is not in a state of thermal equilibrium is because during early times the relatively high energy densities meant that matter was almost in thermal equilibrium at each point in space. Hence spacetime must have been evolving in order to explain the origin of this radiation.

6.2 Temperature Field of the Cosmic Microwave Background

The next question that arises is why is the temperature of the CMB radiation so low; it is interesting to understand how it evolved during the expansion of the universe. For that, one considers equation (56) with a shift in frequency, $\nu \to \beta\nu$, where β is a constant. Thus for the spectrum to maintain its shape, the temperature has to scale as $T \to \frac{T}{\beta}$ such that one can therefore conclude that this effect of keeping the shape unaltered motivates the following relation

$$T \propto \frac{1}{a}. \tag{58}$$

where a is the scale factor. This relation works only for low temperatures, furthermore considerations about including energy from phase transition were omitted [1]. Hence the universe cools down as it expands.

The CMB is not a completely isotropic radiation bath. Divergences, known as fluctuations, are analysed using perturbations. These fluctuations are important in the growth of large scale structures such as stars and galaxies. These divergences are the most important piece of information one can have about the evolution of the universe. High precision cosmology have quantified these fluctuations at the level of one part in a hundred thousand. The following figure depicts the fluctuations in the temperature field.

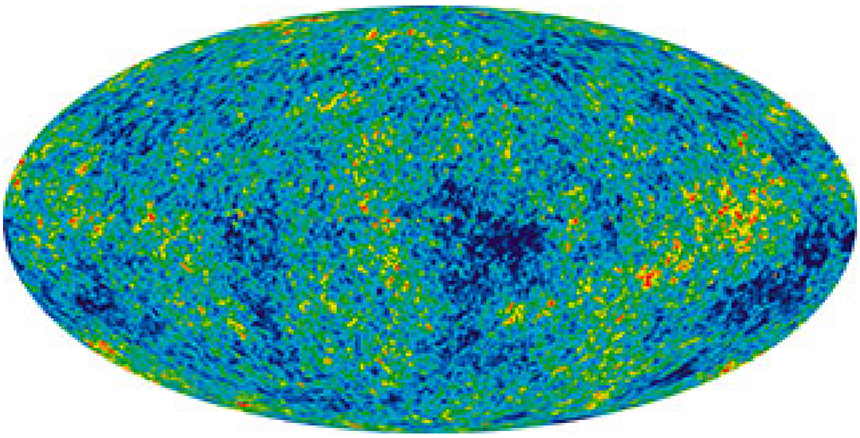

Figure 4: The picture of the early universe obtained from WMAP nine year survey showing temperature anisotropies that later leads to growth of large scale structures of the observable universe after perturbations re-enter the horizon. The acausal correlation and small temperature imperfectness of the radiation bath are consistent with an inflationary scenario. (Credit: NASA/ WMAP Science Team.)

The temperature anisotropies are studied by expanding the temperature difference, ΔT in spherical harmonics, where T_0 is the temperature of the CMB today. Similarly to three dimensional Fourier transforms the signals due to different angular scale contributions are therefore expanded via

$$\frac{\Delta T}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi), \tag{59}$$

where a_{lm} are the expansion coefficients, θ and ϕ are spherical polar angles in the sky and the sum is defined for values of $l = 0, 1, 2, ..\infty$ and $m = -l, .., 0, .., l$ such that there are $2l + 1$ values of m for each value of l. A quantity that is of great interest is the angular power spectrum which is also defined in the context of spherical harmonic monopoles [1] (Ignoring complex quantities)

$$C_l = \langle |a_{lm}|^2 \rangle, \tag{60}$$

where a_{lm} are independent random variables which allows one to define the variance. This is because the CMB sky is a Gaussian random field, thus the power spectrum can be fully obtained by this variance of the amplitudes of the multipole components $l(l+1)C_l$ against the multipole moment, l. Thus for $l \geqq 1$, one can plot the CMB power spectrum is as shown below

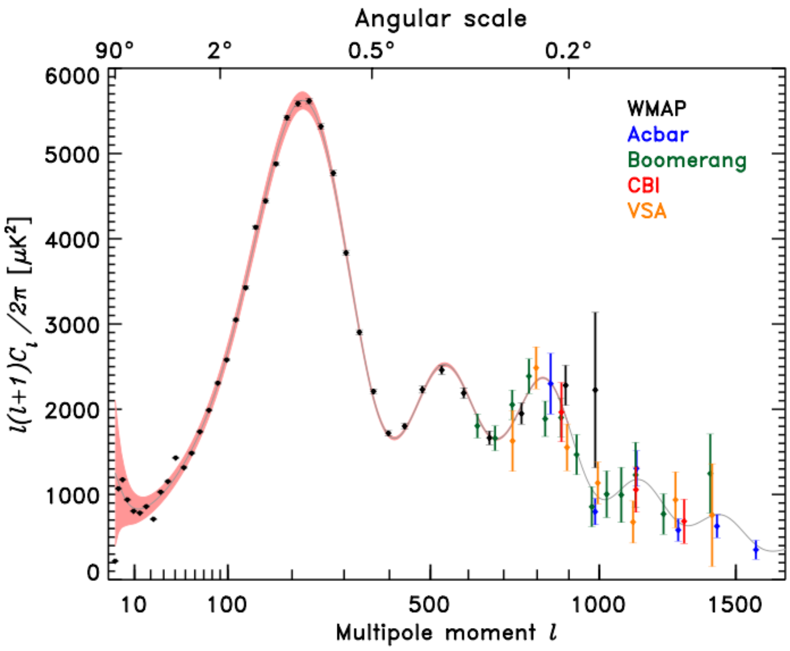

Figure 5: The CMB power spectrum in terms of multipole moments (or angular scales). As resolution decreases it can be seen how the data becomes more accurate. The successive peaks are much less spaced out which perfectly corrolates with the fact that increasing l increases accuracy of the spectrum. The solid line represents the best fit model. [Credit: From experiments in a chronological order - Very Small Array (VSA 2004), Cosmic Background Imager (CBI 2004), Acbar (2004), Boomerang (2005), WMAP (2006).]

From the above figure one can note the large inaccuracy of the data for large angular resolution, that is for small l. This is known as the cosmic variance [1, 2] and is due to the difference of values between the predicted theoretical angular spectrum and the observed experimental angular spectrum. This uncertainty limits cosmological observations and cannot be further reduced because there is only one universe from which the data can be obtained, ours. Thus, using Ergodic theorem [2], one can consider the ensemble average of the temperature field over a long period of time. However since the universe has been evolving for almost 13.7 billion years, one would have to wait a very long time. Therefore the solution is to simply continue improving high precision cosmology measuring techniques and data analysis. The oscillations of present in the spectrum are due to acoustic oscillations produced from rarefactions and compressions of the baryon-photon plasma. The modes of oscillations are classified as acoustic since the waves produced travel at the speed of sound [2].

6.3 Analysis of Cosmological Parameters

The analysis of the power spectrum allows one to understand some important cosmological parameters which help constraining inflationary models that are developed. Such analysis is fully understood using first order (linear) perturbation theory. This process is very long hence only the main results are considered in this section. A detailed explanation of the theory of perturbation can be found in Ref. [2]. The power spectrum is therefore characterized by the spectral index and the tensor-to-scalar ratio which are respectively denoted by n_s and r. If $n = 1$ the spectrum is scale invariant, if n is not equal to 1 the spectrum to be tilted. The spectrum is red if $n < l$ and blue if $n > 1$. The spectral index is defined in terms of the slow roll parameters as

$$n_s = 1 - 6\epsilon - 2\eta, \tag{61}$$

where ϵ and η are given from previously derived equations (28) and (29). The value of n_s given in measured by Planck + WP [21, 22] reads

$$n_s = 0.9603 \pm 0.0073. \tag{62}$$

It can be noted that this is a tight bound on the spectral index which can be used to ruled out numerous models. The calculated values for the spectral indices for chaotic and standard model Higgs inflation are respectively [21, 22, 27] 0.96 and 0.97 which are very close to the experimental value.

Up to this point, the models were normalized by the CMB that is the power spectrum was fixed solely by the amplitude of the cosmic microwave background anisotropies which arised from a study of scalar (density) perturbations. In order to further constraint the models, it important to consider tensor (primordial gravity wave) perturbations. This normalization of the gravitational wave amplitude is incorporated in the tensor-to-scalar ratio, r. The latter is employed in the derivation of the Lyth bound [30].

6.3.1 The Lyth Bound

The tensor-to-scalar ratio is given as

$$r \simeq 16\epsilon, \tag{63}$$

where ϵ is the slow roll parameter defined in equation (28). The interesting thing is that value of r that enables the detection of gravitational waves in future CMB observations is measurable. This tensor-to-scalar ratio can be related to number of e-folds, $\mathcal{N}$ using equations (28) and (30) such that

$$\frac{\Delta\phi}{M_{Pl}} = \int_0^{\mathcal{N}} d\mathcal{N}\sqrt{\frac{r}{8}}, \tag{64}$$

where the limits 0 and $\mathcal{N}$ corresponds to the end of the inflationary era and the moment the anisotropies of the CMB left the horizon. Hence the bound imposed on the inflationary models is the Lyth bound [30],

$$\frac{\Delta\phi}{M_{Pl}} = \mathcal{O}(1) \times \sqrt{\frac{r}{0.01}}, \tag{65}$$

as expressed explicitly in Ref. [16]. The above relation is obtained by expressing in terms of the slow-roll parameter ϵ with higher order corrections and substituting in the potentials and its derivatives in the first order and the higher order terms. The integral is accordingly expanded considering only the first order term. This brief study of super-Planckian field excursion can therefore be used to classify large field inflation models ($r \geqq 0.01$) and small field inflation models.

Hence from calculations based on the models used in the project, the scalar-to-tensor ratio for the Higgs inflation was evaluated to 0.0038. A more accurate value of $r = 0.0034$ is given in Ref. [27] which studied the standard model Higgs as the inflaton. This enables one to successfully give the mass of the Higgs to be equal to 125 GeV [27] as calculated in section 4. The value of r for ϕ^2 inflation is $\leqq 0.10$ as given in Ref. [22] is also in agreement with theoretical prediction however stronger constraints can favor non-minally coupled models such as the one discussed (the standard model Higgs inflation) [21, 22]. The latter is interesting because it sets new lower bound on the value of the scalar-to-tensor ratio. This can be further explored through sub-Planckian calculations [21].

7 Conclusions and Outlook

In this research, the dynamics of an inflationary universe were analysed. The issues that plagued the standard big bang model were discussed and the conclusion was that these problems would be resolved if the universe went thorough a period of rapid expansion. This inflationary paradigm was then developed and further studied via the large field inflation model (Linde' s chaotic inflation) and the standard model Higgs inflation. It has been noted that both models agrees with the data obtained from high precision cosmological observations of the CMB anisotropy. However further cosmological data is still needed in order to further constraint the inflationary models.

Further considerations that can produce results with greater accuracy is to perform radiative corrections (quantum corrections) of the inflationary models discussed in this project. This is because so far, coupling with fermions were omitted. One of the most favored inflationary models is inflation derived from SuperSymmetric theories (SUSY) [16, 17]. The reason being because scalar field theories may be plagued with ultraviolet divergences (hence the search for higher loops corrections). However supersymmetric inflation theories do not possess such divergences and could be better candidates for a complete theory of the inflationary paradigm.

In this research, the gravitational wave production analysis was not considered during the preheating phase, after inflation. One possible extension which may be the subject of further research is to analyse the gravitational wave spectrum due to preheating from the standard model Higgs coupling (and/or from hybrid inflation model). This is motivated from calculations performed in Ref. [32] which only considers quadratic inflation model with a coupling of almost similar form to equation (52) except that the latter is of an opposite sign.

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